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STABILITY AND STABILIZATION OF NONLINEAR DYNAMICAL SYSTEMS

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ABSTRACT

Exponential and asymptotic stability for a class of nonlinear dynamical systems with uncertainties is investigated. Based on the stability of the nominal system, a class of bounded continuous feedback controllers is constructed. By such a class of controllers, the results guarantee exponential and asymptotic stability of uncertain nonlinear dynamical system. A numerical example is also given to demonstrate the use of the main result.

Index Terms : *Control constraint, feedback control, stability, stabilization, uncertainty, uncertain systems*

Nomenclature

R^n	n-dimensional real space
$R^{n\times m}$	Set of all real n by m matrices
A^T	Transpose of matrix A
$\|A\|$	Induced Euclidean norm of matrix A
$\|x\|$	Euclidean norm of $x \in R^n$
$\nabla_x V(t,x)$	Gradient of smooth scalar function $V(t, x)$
$\|a\|$	Absolute value of a real number a
$B_\rho(0)$	Ball in R^n of radius $\rho > 0$ and center at the origin

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1. INTRODUCTION

In recent decades, the stability problem of nonlinear systems have been extensively studied ([1]-[3] and [4]). It is well known that the study of stability theory of nonlinear dynamical systems is carried out by one of two Lyapunov methods, one is the Lyapunov's linearization method, and the other is the Lyapunov's direct method which concerns with construction of the Lyapunov function. The stability problem has motivated the study of Lyapunov function in both finite ([3], [5] and [6]) and infinite dimensional ([1] and [2]) spaces. Here, the Lyapunov's direct method is used. It is the purpose of this paper to investigate the exponential and asymptotic stabilization for nonlinear dynamical systems with control constraint.

This paper is organized as follows. In section II, a theorem which is a criterion for the exponential and asymptotic stability is proposed. Furthermore, based on this theorem, a bounded and continuous state feedback control is proposed to guarantee the exponential and asymptotic stability. In section III, a numerical example is given to illustrate the use of our main result. Finally, the conclusion follows in section IV.

2. PROBLEM FORMULATION AND MAIN RESULT

Consider a class of uncertain nonlinear dynamical systems described by the following state equations:

$$\begin{aligned} &\dot{x}(t) = f(t,x) + F(t,x)\cdot \phi(t,x,u), \;\; t \geq t_0 \geq 0 \\ &x(t_0) = x_0 \end{aligned} \tag{1}$$

where $t \in R_+$ is time, $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, and $\phi(t,x,u)$ represents the system uncertainties. The function, $\phi(\cdot,\cdot,\cdot):[0,\infty)\times R^n \times R^m \to R^m$, $F(\cdot,\cdot):[0,\infty)\times R^n \to R^{n\times m}$, and $f(\cdot,\cdot):[0,\infty)\times R^n \to R^n$, are assumed to be continuous. The corresponding system of (1) without uncertainties, called the nominal system, is described by

$$\begin{aligned} &\dot{x}(t) = f(t,x), \;\; t \geq t_0 \geq 0 \\ &x(t_0) = x_0. \end{aligned} \tag{2}$$

We assume further that the equation (2) has a unique solution corresponding to each initial condition and the origin is the unique equilibrium point. The state feedback controller can be represented by a nonlinear function in the form

$$u(t) = -\gamma(t,x)K^T(t,x).$$

Now, the question is how to synthesize a state feedback controller *u(t)* that can guarantee the asymptotic and exponential stability of nonlinear dynamical system (1) in the presence of uncertainties $\phi(t,x,u)$.

Before giving our synthesis approach, we give some definitions and prove sufficient conditions for the asymptotic and exponential stability of system (2).

Definition 1. The equilibrium zero of (2) is *stable* if, for each ε > 0 and each, $t_o \in R_+$, there exists a $\delta = \delta(\varepsilon, t_0)$ such that $\|x_0\| \le \delta(\varepsilon, t_0)$ implies $\|x(t, x_0)\| \le \varepsilon, \forall t \ge t_0 \ge 0.$

Definition 2. The equilibrium zero of (2) is *attractive* if, for each, $t_0 \in R_+$,there is an $\eta(t_0) > 0$ such that $\|x_0\| \le \eta(t_0)$ implies that the solution $x(t, x_0)$ approaches zero as t approaches infinity.

Definition 3. The equilibrium zero of (2) is *asymptotically stable* if it is stable and attractive.

Definition 4. The equilibrium zero of (2) is *exponentially stable* if there exist positive constants,, ρ, k and γ such that

$$\|x(t, x_0)\| \le k\|x_0\| e^{-\gamma(1-t_0)}, \forall t \ge t_0 = 0, \forall x_0 \in B_\rho.$$

The following theorem provides sufficient conditions for the asymptotic and exponential stability of system (2).

Theorem 1. Assume there exist a sufficiently smooth function $V(t,x)$, positive constants $\lambda_1, \lambda_2, \lambda_3, p$ and q such that, for all and for all $t \ge t_0 \ge 0$ and for all $x(t) \in R^n$

$$\lambda_1 \|x(t)\|^p \le V(t, x(t)) \le \lambda_2 \|x(t)\|^q \tag{3}$$

and the derivative of V along the solution of (2) satisfies

$$\frac{dV(t, x(t))}{dt} = \nabla_t V(t, x(t)) + \nabla_x^T V(t, x(t)) \cdot f(t, x(t)) \le -\lambda_3 \|x(t)\|^q. \tag{4}$$

Then the equilibrium point of the system (2) is asymptotically stable. Moreover, it is exponentially stable if $p = q$.

Proof. Let

$$Q(t, x(t)) = V(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t}. \tag{5}$$

Then, from (5), (4) and (3), we have

$$\begin{aligned} \dot{Q}(t, x(t)) &= \dot{V}(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t} + \frac{\lambda_3}{\lambda_2} V(t, x(t)) e^{\frac{\lambda_3}{\lambda_2} t} \\ &\le -\lambda_3 \|x(t)\|^q e^{\frac{\lambda_3}{\lambda_2} t} + \frac{\lambda_3}{\lambda_2} \lambda_2 \|x(t)\|^q e^{\frac{\lambda_3}{\lambda_2} t} \\ &\le 0. \end{aligned} \tag{6}$$

Integrate both sides of (6), we have, for all $t \geq t_0 \geq 0$

$$Q(t,x(t)) \leq Q(t_0,x(t_0)) = V(t_0,x(t_0,x(t_0))e^{\frac{\lambda_3}{\lambda_2}t_0} \leq \lambda_2 \|x(t_0)\|^q e^{\frac{\lambda_3}{\lambda_2}t_0}. \quad (7)$$

Hence, it follows from (3), (5), and (7), we get

$$\|x(t)\| \leq \left(\frac{V(t,x(t))}{\lambda_1}\right)^{1/p} = \left(\frac{Q(t,x(t))}{\lambda_1}e^{-\frac{\lambda_3}{\lambda_2}t}\right)^{1/p} = \left(\frac{\lambda_2\|x(t_0)\|^q}{\lambda_1}e^{-\frac{\lambda_3}{\lambda_2}(t-t_0)}\right)^{1/p} \quad (8)$$

$$= k\|x(t_0)\|^{q/p}e^{-\gamma(t-t_0)}$$

where $k = \left(\frac{\lambda_2}{\lambda_1}\right)^{1/p}, \gamma = \frac{\lambda_3}{\lambda_2 p}$. Let $\varepsilon > 0$ be given and $\delta(\varepsilon,t_0) = \left(\frac{\varepsilon}{k}\right)^{p/q}$ then whenever $\|x_0\| \leq \delta(\varepsilon,t_0)$ we have

$$\|x(t)\| \leq k\frac{\varepsilon}{k}e^{-\gamma(t-t_0)} < \varepsilon, \ \forall t \geq t_0 \geq 0.$$

Therefore, the equilibrium zero of (2) is stable. Moreover, one can easily see that the right-hand side of (8) approaches zero when t approaches infinity. Hence, the equilibrium zero of (2) is attractive and therefore asymptotically stable. In particular, when $p = q$ the inequality (8) becomes

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\gamma(t-t_0)}, \ \forall t \geq t_0 \geq 0.$$

that is the equilibrium zero of (2) is exponentially stable.

We shall use Theorem 1 to fine the condition on *u(t)* that can guarantee the asymptotic and exponential stability of nonlinear dynamical system (1). Let us introduce for system (1) the following assumptions:

(B1) The components of the control vector are physically limited by

$$|u| < c_i, \ \forall i = 1,2,\ldots,m \quad (9)$$

with $c_i > 0, \quad \forall i = 1,2,\ldots,m.$

(B2) There exist a sufficiently smooth function *W(t,x)*, positive constants λ_1, λ_2, p and q such that for all $x \in R^n$, for all $t \geq t_0 \geq 0$, we have

$$\lambda_1\|x(t)\|^p \leq W(t,x) \leq \lambda_2\|x(t)\|^q \quad (10)$$

and the derivative of W along the solution of the nominal system $\dot{x}(t) = f(t,x)$ satisfies

$$\frac{dW(t,x(t))}{dt} = \nabla_t W(t,x(t)) + \nabla_X^T W(t,x(t)) \cdot f(t,x(t)) \le 0 \tag{11}$$

Remark : The nominal system $\dot{x}(t) = f(t,x)$ is stable with (B2) (See [3] pp. 53-54).

(B3) There exist positive continuous functions $\varepsilon(t,x)$, $f_1(t,x)$, $f_2(t,x)$, $f_3(t,x)$ and positive constants λ_3 and α such that

$$\begin{aligned} y^T \cdot \phi_1(t,x,y) \ge & -f_1(t,x)\|y\| + f_2(t,x)\|y\|^2 - f_3(t,x)\|y\|^3 \\ & + \frac{\lambda_3 f_2(t,x)\|x(t)\|^q}{2f_3(t,x)\left[\|K\| + \varepsilon(t,x)\right]}, \\ & \forall y \in R^m,\ \forall x \in R^n,\ \forall t \ge t_o \ge 0 \end{aligned} \tag{12}$$

where

$$f_2^2(t,x) \ge 4 f_1(t,x) f_3(t,x),\ \forall x \in R^n, \forall t \ge t_0 \ge 0 \tag{13}$$

$$f_1(t,x)\|K\| \le \alpha\|x\|^q,\ \forall x \in R^n, \forall t \ge t_0 \ge 0 \tag{14}$$

$$\phi_1(t,x,y) := \phi(t,x,\frac{2c_1}{\pi}\tan^{-1} y_1 \frac{2c_2}{\pi}\tan^{-1} y_2,\ldots,\frac{2c_m}{\pi}\tan^{-1} y_m), \tag{15}$$

$$y := \left[y_1, y_2, \ldots, y_m\right]^T \in R^m, \tag{16}$$

and

$$K(t,x) := F^T(t,x)\nabla_x W(t,x),\ \forall x \in R^n, \forall t \ge t_0 \ge 0.$$

Lemma 1. Under the assumptions (B2) and (B3), we have

$$f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \alpha\|x(t)\|^q \le 0,\ \forall x \in R^n, \forall t \ge t_0 \ge 0,$$

where

$$\gamma(t,x) := \frac{f_2(t,x)}{2f_3(t,x)\left[\|K\| + \varepsilon(t,x)\right]},\ \forall x \in R^n, \forall t \ge t_0 \ge 0$$

and

$$K(t,x) := F^T(t,x)\nabla_x W(t,x),\quad x \in R^n, \forall t \ge t_0 \ge 0.$$

Proof. $f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \alpha\|x(t)\|^q$

$$= f_1\|K\| - \frac{f_2^2\|K\|^2}{2f_3(\|K\|+\varepsilon)} + \frac{f_2^2\|K\|^3}{4f_3(\|K\|+\varepsilon)^2} - \alpha\|x(t)\|^q$$

$$= \frac{4f_1f_3\|K\|^3 + 4f_1f_3 2\|K\|^2\varepsilon + 4f_1f_3\|K\|\varepsilon^2 - 2f_2^2\|K\|^3 - 2f_2^2\|K\|^2\varepsilon}{4f_3(\|K\|+\varepsilon)^2}$$

$$+ \frac{f_2^2\|K\|^3}{4f_3(\|K\|+\varepsilon)^2} - \alpha\|x(t)\|^q$$

$$= \frac{-\|K\|^3\left[f_2^2 - 4f_1f_3\right] - 2\varepsilon\|K\|^2\left[f_2^2 - 4f_1f_3\right]}{4f_3(\|K\|+\varepsilon)^2} + \frac{4f_1f_3\|K\|\varepsilon^2}{4f_3(\|K\|+\varepsilon)^2} - \alpha\|x(t)\|^q$$

$$= \frac{f_1\|K\|\varepsilon^2 - \alpha\|x(t)\|^q(\|K\|+\varepsilon)^2}{(\|K\|+\varepsilon)^2}$$

$$= \frac{f_1\|K\|\varepsilon^2 - \alpha\|x(t)\|^q\|K\|^2 - 2\alpha\|x(t)\|^q\|K\|\varepsilon - \alpha\|x(t)\|^q\varepsilon^2}{(\|K\|+\varepsilon)^2}$$

$$= \frac{(f_1\|K\| - \alpha\|x(t)\|^q)\varepsilon^2 - \alpha\|x(t)\|^q\|K\|^2 - 2\alpha\|x(t)\|^q\|K\|\varepsilon}{(\|K\|+\varepsilon)^2}$$

$$\leq 0.$$

Theorem 2. The system (1) satisfying the assumptions (B1)-(B3) is asymptotically stable and if $p = q$ it is exponentially stable under the control

$$u_i(t) = \frac{2c_i}{\pi}\tan^{-1}\left[y_i(t)\right], \quad \forall i = 1,2,\ldots,m. \tag{17}$$

Here

$$\left[y_1(t), y_2(t), \ldots, y_m(t)\right] = -\gamma(t,x)K^T(t,x), \tag{18}$$

$$\gamma(t,x) := \frac{f_2(t,x)}{2f_3(t,x)(\|K\| + \varepsilon(t,x))}, \tag{19}$$

and

$$K(t,x) := F^T(t,x)\nabla_x W(t,x), \tag{20}$$

with $\alpha < \lambda_3$.

Proof. By (1) and (15)-(17), one has

$$\begin{aligned}\dot{x}(t) &= f(t,x) + F(t,x)\cdot(\phi(t,x,u_1,u_2,\ldots,u_m)\\ &= f(t,x) + F(t,x)\cdot\phi(t,x,\frac{2c_1}{\pi}\tan^{-1}y_1,\frac{2c_2}{\pi}\tan^{-1}y_2,\ldots,\frac{2c_m}{\pi}\tan^{-1}y_m)\\ &= f(t,x) + F(t,x)\cdot\phi(t,x,y),\ \forall x\in D\subset R^n,\ t\geq t_0\geq 0.\end{aligned}$$

Let W*(t,x)* be a Lyapunov function candidate of (1) with (17)-(20). The time derivative of W*(t,x)* along the trajectories of the closed-loop system, using (B2), is given by

$$\begin{aligned}\dot{W} &= \nabla_t W + \nabla_x^T W\left[f + \cdot\phi_1\right]\\ &\leq \nabla_x^T WF\cdot\phi_1.\end{aligned} \tag{21}$$

From (12) and (18)-(19), we have

$$y^T\cdot\phi_1 \geq -f_1\gamma\|K\| + f_2\gamma^2\|K\|^2 - f_3\gamma^3\|K\|^3 + \lambda_3\gamma\|x(t)\|^q.$$

Multiply both sides by $-\frac{1}{\gamma}$ and from (18), and (20), we have

$$K^T = \cdot\phi_1 = \nabla_x^T WF\cdot\phi_1 \leq f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \lambda_3\|x(t)\|^q. \tag{22}$$

Substitute (22) into (21), we get

$$\begin{aligned}&\leq f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \lambda_3\|x(t)\|^q + \alpha\|x(t)\|^q - \alpha\|x(t)\|^q\\ &= -(\lambda_3-\alpha)\|x(t)\|^q + f_1\|K\| - f_2\gamma\|K\|^2 + f_3\gamma^2\|K\|^3 - \alpha\|x(t)\|^q.\end{aligned} \tag{23}$$

Simplifying (23) by using (19)-(20), we get, by Lemma 1,

$$\dot{W} \leq -(\lambda_3-\alpha)\|x(t)\|^q. \tag{24}$$

By virtue of theorem 1, the proof is completed.

3. EXAMPLE

Consider the following uncertain nonlinear system:

$$\dot{x}(t) = \begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^{\frac{3}{2}}}{2}\end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\cdot\left\{a(t)u + b(t)u^2 + c(t)\tan u - 9\right\} \tag{25}$$

where $\in R$, $x := (x_1, x_2)^T \in R^2$, $-1 \le a(t) \le 1$, $-1 \le b(t) \le 1$, and $4 \le c(t) \le 5$ for all $t \ge t_0 \ge 0$. The coefficients *a(t), b(t),* and *c(t)* are arbitrarily chosen to satisfy (12)-(14). The control u is limited by $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$, and

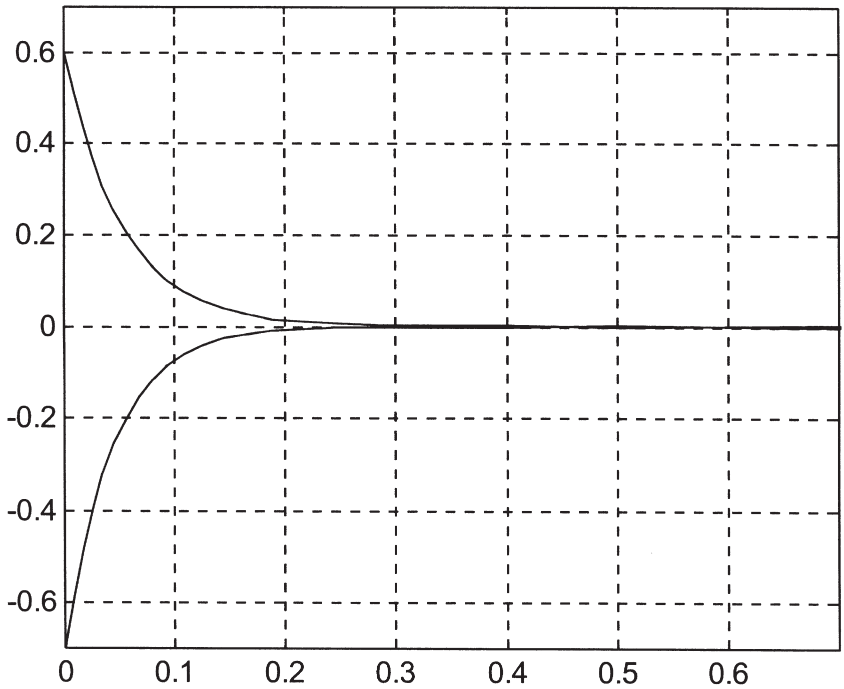

Figure 1 : *The state trajectories of the feedback-controlled system for (25).*

$$f(t,x) = \begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^3}{2} \end{pmatrix}, \; F(t,x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\phi(t,x,u) = \left\{a(t)u + b(t)u^2 + c(t)\tan u - 9\right\}.$$

Choose a positive functional

$$W(t,x) = 2x_1^2 + x_2^2.$$

Then (10) and (11) are satisfied with λ_1=1, λ_2 = 2, *p=2* and *q=2*. In fact,

$$\lambda_1\|x\|^p = x_1^2 + x_2^2 \le W(t,x) = 2x_1^2 + x_2^2 \le 2(x_1^2 + x_2^2) = \lambda_1\|x\|^q$$

and

$$\nabla_x^T W(t,x) f(t,x) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) W(t,x) \cdot f(t,x)$$

$$= (4x_1, 2x_2)\begin{pmatrix} x_2 - x_1^3 \\ -2x_1 - \frac{x_2^3}{2} \end{pmatrix}$$

$$= 4x_1,(x_2 - x_1^3) + 2x_2\left(-2x_1 - \frac{x_2^3}{2}\right)$$

$$= -4x_1^4 - x_2^4 \le 0.$$

From (15), we have

$$\phi_1(t,x,y) := \phi(t,x,\tan^{-1} y)$$

$$= a(t)\tan^{-1} y + b(t)(\tan^{-1} y)^2 + c(t)y - 9.$$

Hence, in (12), we have

$$y^T \cdot \phi_1(t,x,y) = \left[a(t)\tan^{-1} y + b(t)(\tan^{-1} y)^2\right] y + c(t)y^2 - 9y$$

$$\le -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - 9y$$

$$\ge -\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 - 9y.$$

This suggests that in (12) we choose

$$f_1(t,x) = \left(\frac{\pi}{2} + \frac{\pi^2}{4}\right),\ f_2(t,x) = 5,\ and\ f_3(t,x) = 1.$$

It follows that (13) is satisfied. In fact,

$$f_2^2(t,x) = 25 \ge 4 f_1(t,x) f_3(t,x) = 4\left(\frac{\pi}{2} + \frac{\pi^2}{4}\right) \cdot 1$$

$$\approx 16.15.$$

By (20) and (19), with $\varepsilon(t,x) = 1$, we obtain

$$K(t,x) = 4x_1^2 + 2x_2^2,$$

and

$$\gamma(t,x) = \frac{5}{2(4x_1^2 + 2x_2^2 + 1)}.$$

Using (18), (12) becomes

$$y^T \cdot \phi_1(t,x,y) \geq -\left(\frac{\pi}{2}+\frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + 9\gamma K$$
$$= -\left(\frac{\pi}{2}+\frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + \frac{9\cdot 5(4x_1^2+2x_2^2)}{2(4x_1^2+2x_2^2+1)}$$
$$\geq -\left(\frac{\pi}{2}+\frac{\pi^2}{4}\right)|y| + 5|y|^2 - |y|^3 + \frac{9\cdot 5\cdot 2(x_1^2+x_2^2)}{2(4x_1^2+2x_2^2+1)}.$$

Thus (12) holds with $\lambda_3 = 18$. Choosing $\alpha = 17 \leq \lambda_3$, such that (14) holds, i.e.,

$$f_1\|K\| \approx 4.03(4x_1^2+2x_2^2) \leq 17(x_1^2+x_2^2) = \alpha\|x\|^q.$$

Finally, owing to (17) and (18), it can be obtained that

$$u(t) = \tan^{-1}[y(t)]$$

By Theorem 2, we conclude that (25) with the bounded control (26) is exponentially stable. With $a(t) = b(t) = 1$, $c(t) = 5$, $x_1(0) = -0.70$, $x_2(0) = 0.60$, the state trajectories of the feedback-controlled system is depicted in Fig. 1. It can be seen from equation (26) that $u(t)$ is bounded by $-\frac{\pi}{2} < u(t) < \frac{\pi}{2}$.

4. CONCLUSION

In this paper, the exponential and asymptotic stabilization of nonlinear dynamical systems with control constraint has been considered. A bounded and continuous state feedback control for the exponential and asymptotic stability for the closed-loop system is proposed. Finally, a numerical example has also been given to demonstrate the use of our main result.

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