RELAXED CONTROL FOR A CLASS OF SEMILINEAR IMPULSIVE EVOLUTION EQUATIONS[1]

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ABSTRACT

Relaxed control for a class of semilinear impulsive evolution equations is investigated. Boundedness of solutions of semilinear impulsive evolution equations is proved and properties of original and relaxed trajectories are discussed. The existence of optimal relaxed control and relaxation results are also presented.

Keywords: Impulsive system; banach space; semilinear equation; relaxation

1. INTRODUCTION

In this paper, we present sufficient conditions of optimality for optimal relaxed control problems arising in systems governed by semilinear impulsive evolution equations on Banach spaces. The general descriptions of such systems are given below.

$$\frac{d}{dt}x(t) = Ax(t) + f(t, x(t), u(t)),\ t \in I \setminus D \tag{1a}$$

$$x(0) = x_0 \in X, \tag{1b}$$

$$\Delta x(t_i) = F_i(x(t_i)),\ i = 1, 2, ..., n, \tag{1c}$$

where $I \equiv [0, T]$ is a bounded closed interval of the real line R, and let the set $D \equiv \{t_1, t_2, ..., t_n\}$ be a partition on $[0, T]$ such that $0 < t_1 < t_2 < ... < t_n < T$. In general, the operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$, f is a nonlinear perturbation, $\Delta x(t) \equiv x(t_i^+) - x(t_i^-) \equiv x(t_i^+) - x(t_i)$, $i = 1, 2, ...n$, and $F_i's$ are nonlinear operators. This model includes all the standard models used by many authors in the field (see Sattayatham & Huawu [8], Ahmed [1]). The objective functional is given by $J(x, u) = \int_0^T L(t, x(t), u(t))dt$.

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In recent years impulsive evolution equations on infinite dimensional Banach spaces have been considered in several papers by Liu [5] and Ahmed [1]. Liu considers the problem of existence and regularity of the solution while Ahmed considers the optimal impulsive control problem and necessary conditions, but sufficient conditions of relaxation for optimality were not addressed. We wish to present just that. Before we can consider such problems, we need some preparation. The rest of the paper is organized as follows. In Section 2, some basic notations and terminology are presented. Section 3 contains some preparatory results. Relaxed impulsive systems are presented in Section 4. Sufficient conditions of relaxation for optimality are discussed in Section 5.

2. SYSTEM DESCRIPTIONS

Let X be a Banach space. Let $C([0, T], X)$ be the Banach space of all continuous functions from $[0, T]$ into X with the supremum norm, i.e., $\|x\| = \sup\{\|x(t)\|_x : 0 \le t \le T\}$. The operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \ge 0\}$.

Let $PC([0, T], X)$ denote the space of piecewise continuous functions on $[0, T]$ with values in X which are left continuous and possessing right hand limits. Equipped with the supremum norm topology, it is a Banach space. Consider the following evolution systems

$$\frac{d}{dt}x(t) = Ax(t) + g(t, x(t)), \ t \in I \setminus D, \tag{2a}$$

$$x(0) = x_0 \in X, \tag{2b}$$

$$\Delta x(t_i) = F_i(x(t_i)), \ i = 1, 2, ..., n. \tag{2c}$$

By a mild solution $x(t, x_0)$ of the system (2a)-(2c) corresponding to the initial state $x_0 \in X$, we mean a function $x \in PC([0, T], X)$ such that $x(0) = x_0$, and satisfy the following integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)g(s, x(s))ds + \sum_{0<t_i<t} T(t-t_i)F_i(x(t_i)), \ 0 \le t \le T.$$

3. PREPARATORY RESULTS

For the study of relaxation for optimality, it is essential to guarantee the existence and uniqueness of solutions of the impulsive evolution equation and certain other related equations. Here in this section, for the convenience of the reader, we quote some results from the recent work of Liu [4]. But first, we recall some hypotheses on the data of problem (2a)-(2c).

(G) $g : I \times X \to X$ is an operator such that

1. $t \mapsto g(t, x)$ is measurable, the map $x \mapsto g(t, x)$ is continuous, and there exists a constant $L > 0$ such that

$$\|g(t, x) - g(t, y)\| \le L\|x - y\|, \ t \in I, x, y \in X.$$

2. There exists a constant $k > 0$ such that

$$\| g(t, x) \| \leq k (1 + \| x \|), t \in I, x \in X.$$

(F) $F_i : X \to X$, $i = 1, 2, ..., n$, are continuous and there exist constants $h_i > 0$, $i = 1, 2, ..., n$, such that

$$\left\|F_i(x) - F_i(y)\right\| \leq h_i \left\|x - y\right\|, x, y \in X.$$

(A) Let $T(\cdot)$ be the strongly continuous semigroup generated by the unbounded operator A. Let $B(X)$ be the Banach space of all linear and bounded operators on X. Denote

$$M \equiv \max_{t \in [0, T]} \left\|T(t)\right\|_{B(X)},$$

and assume that

$$M[LT + \sum_{k=1}^{n} h_i] < 1.$$

By the uniform boundedness principle $\|T(t)\|_{B(X)}$ is bounded on $[0, T]$, so M in hypothesis (A) is finite. We state the following results which give sufficient conditions for the existence of a mild solution.

Theorem 3.1: Let hypotheses (A), (G), and (F) be satisfied. Then for every $x_0 \in X$, equations (2a)-(2c) has a unique mild solution. Moreover, the set of mild solutions is bounded in $PC([0, T], X)$.

Proof: For the existence and uniqueness of the mild solutions, see Liu ([5], Theorem 2.1). Moreover, one can prove the boundness of the set of mild solutions by using Grownwall inequality. To see this, suppose that $x(\cdot)$ is a mild solution of equation (2a)-(2c). Then we have

$$\begin{aligned}\| x(t) \| &\leq \| T(t)x_0 \| + \int_0^t \| T(t-s) \| \, \| g(s, x(s)) \| ds + \sum_{0<t_i<t} \| T(t-t_i) \| \, \| F_i(x(t_i)) \| \\ &\leq M \| x_0 \| + Mk \int_0^t (1 + \| x(s) \|) ds + M \sum_{0<t_i<t} \| F_i(x(t_i)) \| \\ &\leq M \| x_0 \| + MkT + Mk \int_0^t \| x(s) \| ds + M_1.\end{aligned}$$

Applying Grownwall inequality on each subinterval for which $x(t)$ is continuous, we obtain

$$\begin{aligned}\| x(t) \| &\leq (M_1 + M \| x_0 \| + MkT) e^{Mk \int_0^t d\tau} \\ &\leq (M_1 + M \| x_0 \| + MkT) e^{MkT} \equiv M_2,\end{aligned}$$

for some constants M_1 and M_2. This proves that the set of mild solutions is bounded in $PC([0, T], X)$.

Now, let us consider the corresponding control system. We model the control space by a separable complete metric space Z (i.e., a Polish space). By P_f (P_{fc}), we denote a class of nonempty closed (closed and convex) subsets of Z. Let $I = [0, T]$. Recall that a multifunction $\Gamma : I \to P_f(Z)$ is said to be measurable if for each $F \in P_f(Z)$, $\Gamma^{-1}(F)$ is Lebesgue measurable in I. We defined S_Γ to be the set of all measurable selections of $\Gamma(\cdot)$, i.e.,

$S_\Gamma = \{u : I \to Z \mid u(t)$ is measurable and $u(t) \in \Gamma(t)$, μ-a.e. $t \in I\}$,

where μ is the Lebesgue measure on I. Note that the set $S_\Gamma \neq \phi$ if $\Gamma: I \to P_f(Z)$ is measurable (see Li & Yong [6], Theorem 2.23, p.100). Consider the following control systems

$$\frac{d}{dt}x(t) = Ax(t) + g(t, x(t), u(t)),\ t \in [0, T] \setminus D, \tag{3}$$

$$x(0) = x_0 \in X,$$

$$\Delta x(t_i) = F_i(x(t_i)),\quad i = 1, 2, \dots, n.$$

Here, we require the operators A, and $F_i's$ of (7) to satisfy hypothesis (A) and (F) respectively. We now give some new hypotheses for the remaining data.

(U) $U : I \to P_{fc}(Z)$ is a measurable multifunction satisfying $U(\cdot) \subset K$, where K is a compact subset of Z. For the admissible controls, we choose the set $U_{ad} = S_U$.

(G1) $g : I \times X \times Z \to X$ is an operator such that

1. $t \mapsto g(t, x, z)$ is measurable, the map $(x, z) \mapsto g(t, x, z)$ is continuous on $X \times Z$, and there is a constant $L > 0$ such that

$$\left\|g(t, x_1, z) - g(t, x_2, z)\right\| \le L\left\|x_1 - x_2\right\|,\quad \text{for all}\quad t \in I,\ x_1, x_2 \in X,\ \text{and}\ z \in Z.$$

2. There exists a constant $k > 0$ such that $\|g(t, x, z)\| \le k(1 + \|x\|)$, $t \in I$, $x \in X$, and $z \in Z$.

By assumption (U), the control set S_U is nonempty and is called the class of original control. Now, let us define

$X_0 = \{(x \in PC([0, T], X) \mid x$ is a solution of (3) corresponding to $u\}$.

X_0 is called the class of original trajectories.

$A_0 = \{(x, u) \in PC([0, T], X) \times S_U \mid x$ is a solution of (3) corresponding to $u\}$.

A_0 is called the class of admissible state-control pairs.

The following theorem guarantees that $X_0 \neq \phi$. Its proof follows immediately from Theorem 3.1 by defining the function $g_u(t, x) = g(t, x, u)$ and noting that g_u satisfies all hypotheses of Theorem 3.1.

Theorem 3.2: Assume that hypotheses (A),(F), (G1) and (U) hold. For every $u \in S_U$, equation (3) has a unique mild solution $x(u) \in PC([0, T], X)$. Moreover, the set of mild solutions is bounded in $PC([0, T], X)$.

4. RELAXED IMPULSIVE SYSTEMS

We consider the following optimal control problem

$$\text{(P)}\qquad \inf\{J(x, u) = \int_0^T L(t, x(t), u(t))dt\}$$

subject to equation (3).

It is well known that, to solve optimization problems involving (P) and obtain an optimal state-control pair, we need some kind of convexity hypothesis on the orientor field $L(t, x(t), u(t))$. If the convexity hypothesis is no longer satisfied, in order to get an optimal admissible pair, we need to pass to a larger system with measure control (or know as "relaxed control") in which the orientor field has been convexified. For this purpose, we introduce the relaxed control and the corresponding relaxed systems.

Let Z be a separable complete metric space (i.e., a Polish space) and $B(Z)$ be its Borel σ-field. Let (Ω, Σ, μ) be a measure space. We will denote the space of probability measures on the measurable space $(Z, B(Z))$ by $M_+^1(Z)$.

A Caratheodory integrand on $\Omega \times Z$ is a function $f : \Omega \times Z \to R$ such that $f(\cdot, x)$ is Σ-measurable on Ω, $f(\omega, \cdot)$ is continuous on Z for all $\omega \in \Omega$, and $\sup\{|f(\omega, z)| : z \in Z\} \leq \alpha(\omega)$, a.e., for some functions $\alpha(\cdot) \in L_1(\Omega)$. We denote the set of all Caratheodory integrands on $\Omega \times Z$ by $Car(\Omega, Z)$.

By a transition probability, we mean a function $\lambda : (\Omega \times B(Z) \to [0, 1]$ such that for every $A \in B(Z)$, $\lambda(\cdot, A)$ is Σ-measurable and for every $\omega \in \Omega$, $\lambda(\omega, \cdot) \in M_+^1(Z)$. We use $R(\Omega, Z)$ to denote the set of all transition probabilities from (Ω, Σ) into $(Z, B(Z))$. Following Balder [2], we can define a topology on $R(\Omega, Z)$ as follows : Let $f \in Car(\Omega, Z)$ and define

$$I_f(\lambda) = \int_\Omega \int_Z f(\omega, z)\lambda(\omega)(dz)d\mu(\omega). \tag{4}$$

The weak topology on $R(\Omega, Z)$ is defined as the weakest topology for which all functional $I_f : R(\Omega, Z) \to R, f \in Car(\Omega, Z)$, are continuous.

Supposing $\Omega = I = [0, T]$ and Z is a compact Polish space, then the space $Car(I, Z)$ can be identified with the separable Banach space $L_1(I, C(Z))$ where $C(Z)$ is the space of all real valued continuous functions on Z. To see this, we associate to each Caratheodory integrand $\phi(\cdot, \cdot)$ the map $t \mapsto \phi(t, \cdot) \in C(Z)$. Let $M(Z)$ be the space of all regular bounded countably additive measures defined on $B(Z)$. We note that $M(Z)$ is a Banach space under the total variation norm, i.e., $\| \lambda \|_{M(Z)} = | \lambda | (Z)$. Then by the Riesz representation theorem, the dual $[C(Z)]^*$ can be identified algebraically and metrically with $M(Z)$. The duality pair between $M(Z)$ and $C(Z)$ is given by

$$\langle \lambda, f \rangle = \int_Z f(z)\lambda(dz).$$

So $M(Z)$ is a separable (see Warga [9], p.265) dual Banach space and hence has a Radon-Nikodym property. This observation combined with Theorem 1 of Diestel and Uhr [3, p. 98], tells us that

$$L_1(I, C(Z))^* = L_\infty(I, M(Z)). \tag{5}$$

Hence the weak topology on $R(I, Z)$ coincides with the relative $w^*(L_\infty(I, M(Z)))$, $L_1(I, C(Z))$-topology.

The duality pair between $L_\infty(I, M(Z))$ and $L_1(I, C(Z))$ is given by

$$\langle\langle \lambda, f \rangle\rangle = \int_0^T \langle \lambda(t), f(t) \rangle dt \tag{6}$$

$$= \int_0^T \int_Z f(t)(z)\lambda(t)(dz)dt$$

$$= \int_0^T \int_Z f(t, z)\lambda(t)(dz)dt,$$

which is the same formula as in (4) with $f(t, z) \equiv f(t)(z)$. This fact will be useful in the study of the relaxed control system where the control functions are transition probabilities.

Now we introduce some assumptions imposed on the class of relaxed controls which will be denoted by S_Σ.

(U1) Z is a compact Polish space, $U : I \to P_{fc}(Z)$ is a measurable multifunction.

Define $\Sigma(t) = \{\lambda \in M_+^1(Z), \lambda(U(t)) = 1\}$ and let $S_\Sigma \subseteq R(I, Z)$ be the set of transition probabilities on $I \times B(Z)$ that are measurable selections of $\Sigma(\cdot)$. For any $u \in S_U$, we define the relaxation $\delta_u \in S_\Sigma$ of u by $\delta_u(t) \equiv$ Dirac probability measure at $u(t)$. Then we can identify $S_U \subseteq S_\Sigma$. From now on, we shall consider S_U and S_Σ as a subspace of the topological space $R(I, Z)$ with the weak topology defined above.

We list two lemmas which will be useful in discussing the relaxation problem. The proofs can be found in Warga [9, Theorem IV 2.1] and Balder [2, Corollary 3] respectively.

Lemma 4.1: Suppose Z is a compact Polish space. Then S_Σ is convex, compact, and sequentially compact.

Lemma 4.2: S_U is dense in S_Σ.

The following theorem is the Arzela-Ascoli Theorem for continuous vector-valued functions. A proof of this result can be found in Carroll [3, Theorem 8.18, p. 34].

Theorem 4.3: (Arzela-Ascoli) A subset $K \subseteq C(I, H)$ is relatively compact if and only if K is equicontinuous and for all $t \in I$, $K(t) = \{f(t) \mid f \in K\}$ is a relatively compact subset of H.

Next, let us consider this new larger system know as "relaxed impulsive system"

$$\dot{x}(t) = Ax(t) + \int_Z g(t, x(t), z)\lambda(t)(dz),\ 0 \le t \le T, \tag{7}$$
$$x(0) = x_0,$$
$$\Delta x(t_i) = F_i(x(t_i)), \quad i = 1, 2, ..., n.$$

We will denote the set of trajectories of (7) by X_r, i.e.,

$X_r = \{x \in PC(I, X) \mid x$ is a mild solution of (11) corresponding to $\lambda \in S_\Sigma\}$.

Moreover, the set of admissible state-control pairs of (7) will be denoted by

$A_r = \{(x, \lambda) \in PC(I, X) \times S_\Sigma \mid x$ is a mild solution of (7) corresponding to $\lambda \in S_\Sigma\}$.

Note that $X_0 \subseteq X_r$, since $S_U \subseteq S_\Sigma$, and, if the hypotheses of Theorem 3.2 are satisfied, $X_0 \neq \phi \Rightarrow X_r \neq \phi$. To see this, given any relaxed control $\lambda \in S_\Sigma$, if we set $\bar{g}(t, x(t), \lambda(t)) = \int_Z g(t, x(t), z)\lambda(t)(dz)$ then, working as in the proof of Theorem 3.2, one can show that there exists a relaxed admissible trajectory $x(\lambda)$ corresponding to λ. We now summarize the above discussion into a theorem.

Theorem 4.4: Assume that hypotheses (A), (F), (G1) and (U1) hold. For every $\lambda \in S_\Sigma$, equation (7) has a unique mild solution $x(\lambda) \in PC(I, X)$. Moreover the set X_r is bounded in $PC(I, X)$. i.e., $\| x(\lambda) \| x(\lambda) \|_{PC(I, X)} \leq M$ for all $\lambda \in S_\Sigma$.

The next theorem gives us a useful relation between X_0 and X_r.

Theorem 4.5: If assumptions (A), (G), (F), (G1) and (U1) hold, then $X_r = \overline{X_0}$ (closure is taken in $PC(I, X)$).

Before proving this theorem, we need a lemma.

Lemma 4.6: If assumptions (A), (G), (F), (G1) and (U1) hold and $\lambda_k \to \lambda$ in $R(I, Z)$. Suppose that $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$. Assume further that there exists $y \in PC(I, X)$. such that $x_k \to y$ as $k \to \infty$. Then y is a solution of (7) corresponding to the control variable λ.

Proof: Since x_k is a solution of (7) corresponding to the control variable λ_k then

$$x_k(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds + \sum_{0<t_i<t} T(t-t_i)F_i(x_k(t_i)), 0 \leq t \leq T.$$

We aim to prove that y is a solution of (7) corresponding to the control variable λ, i.e., we shall show that

$$y(t) = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, y(s), z)\lambda(s)(dz)ds + \sum_{0<t_i<t} T(t-t_i)F_i(y(t_i)), 0 \leq t \leq T.$$

For each fixed $0 \leq t \leq T$, and $h^* \in X^*$, we denote the duality pair between X and X^* by $\langle ., . \rangle$ and denote $h_t^*(s, z) \equiv \langle T(t-s)g(s, y(s), z), h^* \rangle$, where $0 \leq s \leq t \leq T$, $z \in Z$. It follows from (G1) that $h_t^*(s, z)$ is a Carathedory integrand. Then, by the topology on $R(I, Z)$, we have

$$\int_{[0,t]} \int_Z \langle T(t-s)g(s, y(s), z), h^* \rangle \lambda_k(s)(dz)dt \to \int_{[0,t]} \int_Z \langle T(t-s)g(s, y(s), z, h^* \rangle \lambda(s)(dz)dt \text{ in } R,$$

as $n \to \infty$. Hence

$$\langle \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)dt, h^* \rangle \to \langle \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda(s)(dz)dt, h^* \rangle \text{ in } R,$$

as $n \to \infty$. Since h^* is an arbitrary element in X^* then

$$\int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)dt \to \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda(s)(dz)dt \text{ in } X, \tag{8}$$

as $n \to \infty$. Moreover, we note that

$$\| \int_0^t T(t-s) \int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)dt$$

$$+ \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)dt - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda(s)(dz)dt \|$$

$$\leq \| \int_0^t T(t-s) \int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)ds \| \tag{9}$$

$$+\| \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda(s)(dz)ds \|$$

It follows from equation (8) that the second expression of inequality (9) converges to zero as $k \to \infty$. The first expression also converges to zero. To see this, we note that

$$\| \int_0^t T(t-s) \int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds - \int_{[0,t]} \int_Z T(t-s)g(s, y(s), z)\lambda_k(s)(dz)ds \|_X$$

$$\leq \int_{[0,t]} \int_Z \| T(t-s) \|_{B(X)} \| g(s, x_k(s), z) - g(s, y(s), z) \|_X \lambda_k(s)(dz)ds$$

$$\leq \int_{[0,t]} \int_Z ML \| x_k(s) - y(s) \|_X \lambda_k(s)(dz)ds \qquad \text{(assumption G1)}$$

$$\leq \int_{[0,t]} \| x_k(s) - y(s) \|_X \left(\int_Z ML\lambda_k(s)(dz)\right)ds$$

$$\leq \int_{[0,t]} \| x_k(s) - y(s) \|_X ML\lambda_k(s)(Z)ds$$

$$\leq ML \int_{[0,t]} \| x_k(s) - y(s) \|_X ds \to 0 \text{ as } k \to \infty. \qquad (\lambda_k(s) \text{ is a probability measure on } Z)$$

Now we turn to the jump part.

$$\| \sum_{0<t_i<t} T(t-t_i)F_i(x_k(t_i)) - \sum_{0<t_i<t} T(t-t_i)F_i(y(t_i)) \|_X$$

$$\leq \sum_{0<t_i<t} \| T(t-t_i) \|_{B(X)} \| F_i(x_k(t_i)) - F_i(y(t_i)) \|_X$$

$$\leq M \sum_{0<t_i<t} \| F_i(x_k(t_i)) - F_i(y(t_i)) \|_X$$

$$\leq M \sum_{0<t_i<t} h_i \| x_k(t_i) - y(t_i) \|_X \to 0 \text{ as } k \to \infty.$$

This prove that

$$\lim_{k\to\infty} x_k = T(t)x_0 + \int_0^t T(t-s) \int_Z g(s, y(s), z)\lambda(s)(dz)ds + \sum_{0<t_i<t} T(t-t_i)F_i(x_k(t_i)).$$

By the assumption, we get $\lim_{k\to\infty} x_k = y$. Hence y is the solution (7) as required.

Lemma 4.7: If assumptions (A), (G), (F), (G1) and (U1) hold, the semigroup $\{T\{t\}\}$ in the assumption (A) is compact, and $\lambda_k \to \lambda$ in $R(I, Z)$. Suppose that $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$, by working with a subsequence is necessary, $x_k \to x$ in $PC(I, X)$ as $k \to \infty$.

Proof: Suppose that $\lambda_k \to \lambda$ in $R(I, Z)$ as $k \to \infty$ and $\{x_k, x\}$ is the solution of (7) corresponding to $\{\lambda_k, \lambda\}$. Since $(x_k, \lambda_k) \in A_r$ for each positive integer k, then (x_k, λ_k) must satisfy the equation

$$x_k(t) = T(t)x_0 + \int_0^t T(t-s)\int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds + \sum_{0<t_i<t} T(t-t_i)F_i(x_k(t_i)), 0 \le t \le T,$$

while (x, λ) satisfies

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\int_Z g(s, x(s), z)\lambda(s)(dz)ds + \sum_{0<t_i<t} T(t-t_i)F_i(x(t_i)), 0 \le t \le T.$$

To finish the proof, we try to choose $y \in X_r$ such that y is a solution of (7) corresponding to this λ and $x_k \to y$ in $PC(I, H)$ as $k \to \infty$. The unique property of solution of (7) implies $x = y$ and hence $x_k \to x$ in $PC(I, X)$ and we are done. Since x_k is a mild solution of (7), then by Theorem 4.4, $\{x_k\}$ is a bounded sequence in $PC(I, X)$. By using the same technique as in [7, p. 193], one can prove that the set $\{x_k\}$ is equicontinuous. To see this, let ρ, t, $t' \in [0, T]$ be such that $0 < \rho < t < t'$. Let $N \equiv \sup\{\| g(s, x_k(s), z)\| : s \in [0, T], z \in Z\}$ which is finite and independent of k since g is continuous on the compact space Z the solution set $\{x_k\}$ is bounded, and hypothesis G_1. Then

$$\begin{aligned}
\| x_k(t') - x_k(t) \| \le\ & \| T(t')x_0 - T(t)x_0 \| + \| \int_t^{t'} T(t'-s)_Z\, g(s, x_k(s), z)\lambda_k(s)(dz)ds \| \\
& + \| (\int_0^{t-\rho} + \int_{t-\rho}^{t})(T(t'-s) - T(t-s)) \int_Z g(s, x_k(s), z)\lambda_k(s)(dz)ds \| \\
& + \sum_{t<t_i<t'} \| T(t'-t_i) \| \, \| F_i(x_k(t_i)) \| + \sum_{t<t_i<t'} \| T(t'-t_i)T(t-t_i) \| \, \| F_i(x_k(t_i) \| \\
\le\ & \| T(t')x_0 - T(t)x_0 \| + M\int_t^{t'} \| \int_Z g(s, x_k(s), z)\lambda_k(s)(dz) \| ds \\
& + \int_0^{t-\rho} \| T(t'-s) - T(t-s) \| \int_Z \| g(s, x_k(s), z)\lambda_k(s)(dz) \| ds \\
& + \int_{t-\rho}^{t} \| T(t'-s) - T(t-s) \| \int_Z \| g(s, x_k(s), z)\lambda_k(dz) \| ds \\
& + \sum_{t<t_i<t'} M \| F_i(x_k(t_i)) \| + \sum_{0<t_i<t'} \| T(t'-t_i) - T(t-t_i) \| \, \| F_i(x_k(t_i) \| \\
\le\ & \| T(t')x_0 - T(t)x_0 \| + MN(t'-t) + N\int_0^{t-\rho} \| T(t'-s) - T(t-s) \| ds + 2MN\rho \\
& + \sum_{t<t_i<t'} M \| F_i(x_k(t_i)) \| + \sum_{0<t_i<t'} \| T(t'-t_i) - T(t-t_i) \| \, \| F_i(x_k(t_i) \| . \qquad (10)
\end{aligned}$$

Since $t > \rho > 0$ is arbitrary, and since $T(t)$ is continuous in the uniform operator topology for $t \ge \rho > 0$, the first four terms on right-hand of inequality (10) tend to zero as t tends to t' and ρ tends to zero. Moreover, the two jump terms also tend to zero as t tend to t' since there is no jump in the interval (t, t') if length $|t - t'|$ is small enough. This proves that the set $\{x_k\}$ is equicontinuous.

Let $K^1 \equiv \{x_k^1\}$ be the restriction of the sequence $\{x_k\}$ on the interval $[0, t_1]$, i.e., $x_k^1(t) = x_k(t)$ on $[0, t_1]$ and equal to zero elsewhere. Clear $K(0)= \{x_0\}$ is compact in H. For $0 < \varepsilon < t \leq t_1$, define

$$K_\varepsilon^1(t) = \{T(\varepsilon)x_k^1(t-\varepsilon) : k = 1, 2, ...\}$$

For each $t \in [0, t_1]$, $K^1(t)$ is a bounded subset of H and, by our hypothesis, $T(t)$ is a compact operator for $t > 0$, it follows from the above expression that $K_\varepsilon^1(t)$ is relative compact for $t \in (\varepsilon, t_1)$ Further, by using the same proof as in (10), one can show that

$$\sup\{\| x_k^1(t) - T(\varepsilon)x_k^1(t-\varepsilon)\| : k = 1, 2, ...\} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Then the set $K_\varepsilon^1(t)$ can be approximated to an arbitrary degree of accuracy by a relatively compact set. Hence $K^1(t)$ itself is relatively compact. Applying Arzela-Ascoli Theorem, the sequence $\{x_k^1\}$ is relatively compact in $C[0, t_1], X)$. Then there exists a subsequence of $\{x_k^1\}$, again denoted by $\{x_k^1\}$, such that

$$x_k^1 \to y^1 \text{ in } C([0, t_1], X) \text{ as } k \to \infty.$$

Now, let $\{x_k^2\}$ be the restriction of the sequence $\{x_k\}$ on the interval $(t_1, t_2]$, i.e., $x_k^2(t) = x_k(t)$ on $(t_1, t_2]$ and equal to zero elsewhere. By using the same proof as above, there exists a subsequence of $\{x_k^2\}$, again denoted by $\{x_k^2\}$, such that

$$x_k^2 \to y^2 \text{ in } C((t_1, t_2], X) \text{ as } k \to \infty.$$

It is obvious that $y^2(t^+) = \lim_{k\to\infty} x_k(t_1^+)$. Hence y^2 possesses a right hand limit. Continue this process until to the interval $(t_{n-1}, t_n]$. Define a function y on $[0, T]$ as follows:

$$y(t) = \begin{cases} x(0) \text{ if } t = 0 \\ y^i(t) \text{ if } t \in (t_{i-1}, t_i), i = 0, 1, 2, ..., n \end{cases}$$

Then $y \in PC([0, T], H)$ and there is a subsequence of $\{x_k\}$ converges to y. Applying lemma 4.6, we get y is also a solution of (7). By uniqueness of the solution of (7), we get $x = y$. Hence there is a subsequence of $\{x_k\}$ converges to x and we are done.

Proof of Theorem 4.5: Firstly, we shall show that $X_r \subset \overline{X_0}$. Let $x \in X_r$, then there exists $\lambda \in S_\Sigma$ such that $(x, \lambda) \in A_r$. By virtue of the density result as in Lemma 4.2, there exists a sequence $\{u_k\} \subset S_U$ such that in $\delta_{u_k} \to \lambda$ in $R(I, Z)$. Let x_k be the solution of (7) corresponding to u_k. Then we have a sequence $\{(x_k, u_k)\} \subset A_0$. Since for each k, $(x_k, u_k) \in A_0$ then (x_k, u_k) must satisfy the equation

$$\begin{aligned} &\dot{x}_k(t) = Ax_k(t) + \int_Z g(t, x_k(t), z)\delta_{u_k}(t)(dz), \\ &x_k(0) = x_0 \in X, \\ &\Delta x_k(t_i) = F_i(x_k(t_i)), i = 1, 2, ..., n, \ k = 1, 2, 3, ... \\ &(0 < t_1 < t_2 < ... < t_n) \end{aligned}$$

Applying Lemma 4.7, we get $x_k \to x$ in $PC(I, X)$. This proves that $x \in \overline{X_0}$, and hence $X_r \subset \overline{X_0}$. Finally, we will show that X_r is closed in $PC(I, X)$. Let $\{x_k\}$ be a sequence of points in X_r such that $(x_k \to x)$ in $PC(I, X)$ as $k \to \infty$. By definition of X_r, there is a sequence $\{\lambda_k\}$ of points in S_Σ such that $(x_k, \lambda_k) \in A_r$, $k = 1, 2, 3, \ldots$. Lemma 4.1 implies that S_Σ is compact in $R(I, Z)$ under the weak topology. Moreover, $R(I, Z)$-topology coincides with the relative $\omega^*(L_\infty(I, M(Z)), L_1(I, C(Z)))$-topology. Then, by passing to a subsequence if necessary, we may assume that $\lambda_k \to \lambda$ in $R(I, Z)$ as $k \to \infty$. Applying Lemma 4.7, there is $x \in X_r$ such that $x_k \to x$ in $PC(I, X)$ as $k \to \infty$. Hence X_r is closed in $PC(I, X)$ and consequently, $\overline{X_0} \subset \overline{X_r} = X_r$. The proof of Theorem 4.5 is now complete.

The following corollary is an immediate consequence of lemma 4.7.

Corollary 4.8: Under assumption of Theorem 4.5, the function $\lambda \mapsto x(\lambda)$ is continuous from $S_\Sigma \subseteq R(I, Z)$ into $PC(I, X)$.

5. EXISTENCE OF OPTIMAL CONTROLS

Consider the following Lagrange optimal control problem (P_r) : Find a control policy $\bar{\lambda} \in S_\Sigma$, such that it imparts a minimum to the cost functional J given by

$$(P_r) \qquad J(\lambda) \equiv J(x^\lambda, \lambda) \equiv \int_I \int_Z l(t, x^\lambda(t), z)\lambda(t)(dz)dt,$$

where x^λ is the solution of the system (7) corresponding to the control $\lambda \in S_\Sigma$. We form the following hypothesis concerning the integrand $l(.,.,.)$.

(L) $l : I \times H \times Z \to R \cup \{+\infty\}$ is Borel measurable satisfying the following conditions.

1. $(\xi, z) \mapsto l(t, \xi, z)$ is lower semicontinuous on $H \times Z$ for each fixed t.
2. There exist $\psi(t) \in L_1(I, R)$ such that $|l(t, \xi, z)| \le \psi(t)$ for almost all $t \in I$.
3. l maps bounded set into bounded set.

Let $m_r = \inf\{J(\lambda) : \lambda \in S_\Sigma\}$. We have the following theorem on the existence of optimal impulsive control.

Theorem 5.1: Suppose assumptions (A), (F), (G1), (U1), (L) hold and Z is a compact Polish space, then there exists $(\bar{x}, \bar{\lambda}) \in A_r$ such that $J(\bar{x}, \bar{\lambda}) = m_r$.

Before giving the proof of Theorem 5.1, we need a lemma. The proof is similar to Lemma 3.3 in [10].

Lemma 5.2: Let $h : I \times H \times Z \to R$ be such that

1. $t \mapsto (t, x, z)$ is measurable and $(x, z) \mapsto h(t, x, z)$ is continuous.
2. $|h(t, x, z)| \le \psi(t) \in L_1(I)$ for all $(x, z) \in H \times Z$.

If $x_k \to x \in C([0, T], H)$ then

$$\bar{h}_k(\cdot,\cdot) \to \bar{h}(\cdot,\cdot) \text{ in } L_1(I, C(Z))$$

as $k \to \infty$, where $\bar{h}_k(t, z) = h(t, x_k(t), z)$ and $\bar{h}(t, z) = h(t, x(t), z)$.

Proof of Theorem 5.1: If $J(\lambda) = +\infty$ for all $\lambda \in S_\Sigma$, then every control is admissible. Assume $\inf\{J(\lambda) : \lambda \in S_\Sigma\} = m_r < +\infty$. By assumption (L), we have $m_r > -\infty$. Hence m_r is finite. Let $\{\lambda_k\}$ be a minimizing sequence so that $\lim_{k\to\infty} J(\lambda_k) = m_r$. By Lemma 4.1, S_Σ is compact in the topology $R(I, Z)$. Hence, by passing to a subsequence if necessary, we may assume that $\lambda_k \to \bar{\lambda}$ in $R(I, Z)$ as $k \to \infty$. This means that $\lambda_k \underline{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \to \infty$. Let $\{x_k, \bar{x}\}$ be the solution of (7) correspond to $\{\lambda_k, \bar{\lambda}\}$. By Lemma 4.5, we get $x_k \to \bar{x}$ in $PC(I, X)$ and $(\bar{x}, \bar{\lambda}) \in A_r$. Next, we shall prove that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

As before, we identify the space of Caratheodory integrand $Car(I, Z)$ with the separable Banach space $L_1(I, C(Z))$. We note that every lower semicontinuous measurable integrand $l : I \times H \times Z \to R \cup \{+\infty\}$ is the limit of an increasing sequence of Caratheodory integrand $\{l_j\} \in L_1(I, C(Z))$ for each fixed $h \in H$. Thus, there exists an increasing sequence of Caratheodory integrands $\{l_j\} \in L_1(I, C(Z))$ such that

$$l_j(t, \bar{x}(t), z) \uparrow l(t, \bar{x}(t), z) \text{ as } j \to \infty \text{ for all } t \in I, z \in Z.$$

Since $x_k \to \bar{x}$ in $PC(I, X)$, by applying Lemma 5.2 on each subinterval of $[0, T]$, $l_j(t, x_k(t), z) \to l_j(t, x_k(t), z)$ as $k \to \infty$ for almost all $t \in I$ and all $z \in Z$. We note that since $\lambda_k \underline{w^*} \bar{\lambda}$ in $L_\infty(I, M(Z))$ as $k \to \infty$, then

$$\begin{aligned} J(\bar{x}, \bar{\lambda}) &= \langle\langle \bar{\lambda}, l \rangle\rangle = \int_I \int_Z l(t, \bar{x}(t), z)\bar{\lambda}(t)(dz)dt \\ &= \lim_{j\to\infty} \int_I \int_Z l_j(t, \bar{x}(t), z)\bar{\lambda}(t)(dz)dt \\ &= \lim_{j\to\infty}\lim_{k\to\infty} \int_I \int_Z l_j(t, x_k(t), z)\lambda_k(t)(dz)dt \\ &\le \lim_{k\to\infty}\lim_{j\to\infty} \int_I \int_Z l_j(t, x_k(t), z)\lambda_k(t)(dz)dt = m_r. \end{aligned}$$

However, by definition of m_r, it is obvious that $J(\bar{x}, \bar{\lambda}) \ge m_r$. Hence $J(\bar{x}, \bar{\lambda}) \ge m_r$. This implies that $(\bar{x}, \bar{\lambda})$ is an optimal pair.

Remark: If $J_0(x, u) = \int_I l(t, x(t), u(t))dt$ is the cost functional for the original problem and $m = \inf\{J_0(x, u) : u \in U_{ad}\}$. In general we have $m_r \le m$. It is desirable that $m_r = m$, i.e., our relaxation is reasonable. With some stronger conditions on l, i.e., the map $(\xi, \eta, z) \mapsto l(t, \xi, z)$ is continuous and $|l(t, \xi, z)| \le \theta_R(t)$ for all most all $t \in I$ and $\theta_R \in L_1(I)$, one can show that $m_r = m$. The proof is similar to Theorem 4B in [10].

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