Vietnam Journal of Mathematics **35**:3(2007) 1–15

Vietnam Journal
of
MATHEMATICS

A Fractional Black-Scholes Model with Jumps*

P. Sattayatham, A. Intarasit, and A. P. Chaiyasena

School of Mathematics, Suranaree University of Technology
Nakhon Ratchasima, Thailand

Received February 24, 2006
Revised August 09, 2007

Abstract. In this paper,we introduce an approximate approach to a fractional Black-Scholes model with jumps perturbed by fractional noise. Based on a fundamental result on the L^2-approximation of this noise by semimartingles, we prove a convergence of theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a fractional jump model as compared to the classical jump model.

2000 Mathematics Subject Classification: 91B28, 65C50.

Keywords: Black-Scholes, approximate models.

1. Introduction

In some recent papers (see for examples [5, 6]), some fractional Black-Scholes model have been proposed as an improvement of the classical Black-Scholes. Common to these models is that they are driven by a fractional Brownian motion and that some stochastic calculus is created by using, for example, Malliavin calculus or Wick product analysis. Recently, and approximate approach to fractional Black-Scholes model is introduced and investigated in [10]. In this paper we use this approach to study a fractional Black-Scholes model with jumps.

Recall that a fractional Brownian motion B_t^H with Hurst index H, is a centered Guassian process such that its covariance function $R(t,s) = EB_t^H B_s^H$ is given by

*This work was supported by Suranaree University of Technology, 2005.

$$R(t,s) = \frac{1}{2}(|t|^\gamma + |s|^\gamma - |t-s|^\gamma), \text{ where } \gamma = 2H \text{ and } 0 < H < 1.$$

If $H = \frac{1}{2}$, $R(t,s) = \min(t,s)$ and B_t^H is the usual standard Brownian motion. In the case $\frac{1}{2} < H < 1$ the fractional Brownian motion exhibits statistical long range dependency in the sense that $\rho_n := E[B_1^H(B_{n+1}^H - B_n^H] > 0$ for all $n = 1, 2, 3, \ldots$ and $\Sigma_{n=1}^{\infty} \rho_n = \infty$ ([9, page 2]). Hence, in financial modelling, one usually assumes that $H \in (\frac{1}{2}, 1)$. Put $\alpha = H - \frac{1}{2}$.It is known that a fractional Brownian motion B_t^H can be decomposed as follows:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)}[Z_t + \int_0^t (t-s)^\alpha dW_s],$$

where Γ is the gamma function,

$$Z_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha] dW_s,$$

and W_t is a standard Brownian motion. We suppose from now on $0 < \alpha < \frac{1}{2}$. Then Z_t has absolutely continuous trajectories and it is the term $B_t := \int_0^t (t-s)^\alpha dW_s$ that exhibits long range dependence. We will use B_t instead of B_t^H in fractional stochastic calculus. The fractional Black-Scholes model under our consideration is of the form

$$\begin{aligned} dS_t &= S_t(\mu dt + \sigma dB_t), 0 \le t \le T, \\ S(0) &= S_0, \end{aligned} \tag{1}$$

where S_t is the price of a stock, μ, and σ are constants, and B_t as given above.

Now, consider the corresponding approximate model of (1)

$$\begin{aligned} dS_\varepsilon(t) &= S_\varepsilon(t)(\mu dt + \sigma dB_\varepsilon(t)), 0 \le t \le T, \\ S_\varepsilon(0) &= S_0 \text{ (same initial condition as in (1))}, \end{aligned} \tag{2}$$

where $B_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dW(s)$, $\frac{1}{2} < H < 1$. Referring to the main result of Thao [10, Theorem 4.2], the solution $S_\varepsilon(t)$ of equation (2) converges to the solution S_t of (1) in $L^2(\Omega)$ as $\varepsilon \to 0$.

In this paper, we extend the main result of Thao [10] to a fractional Black-Scholes model with jumps. We also prove that the solution of our approximate models converges to the solution of the fractional Black-scholes model with jumps. In summary, this paper is organized as follows: In Sec. 2, we review the definition of the Poisson random measure and some preliminary notions of jump-diffusion processes which mostly come from [2]. In Sec. 3, we follow the general setting of [7, page 143] to consider the stock price model with jumps. In Sec. 4, we discuss an approximate model for a fractional stock-price model with jumps. Finally, we give some simulation examples to show the accuracy of approximations by the fractional Black-Scholes model with jumps as compared to the classical Black-Scholes model with jumps.

2. Poisson Random Measures

A Poisson process $(N(t), t \geq 0)$, with intensity λ, is defined as follows:

$$N(t) = \sum_{n \geq 1} 1_{\{T_n \leq t\}},$$

where $T_n = \sum_{i=1}^{n} \tau_i$ and $\tau_1, \tau_2, \ldots$ is a sequence of independent, identically exponentially distributed random variables (defined on some probability space $(\Omega, \boldsymbol{\mathcal{F}}, P)$) with parameter λ, that is, $P(\tau_1 > t) = e^{-\lambda t}$. $N(t)$ is simply the number of jumps between 0 and t, i.e.,

$$N(t) = \#\{n \geq 1, T_n \in [0, t]\}.$$

Similarly, if $t > s$ then

$$N(t) - N(s) = \#\{n \geq 1, T_n \in (s, t]\}.$$

The jump times $T_1, T_2, \ldots$, form a random configuration of points on $[0, \infty)$ and the Poisson process $N(t)$ counts the number of such points in the interval $[0, t]$. This counting procedure defines a measure N on $[0, \infty) := \mathbb{R}^+$ as follows: For any Borel measurable set $A \subset \mathbb{R}^+$,

$$N(\omega, A) = \#\{n \geq 1, T_n(\omega) \in A\} = \sum_{n \geq 1} 1_A(T_n(\omega)).$$

$N(\omega, \cdot)$ is a positive integer valued measure on Borel subsets of $\mathbb{R}^+$. We note that $N(\cdot, A)$ is finite with probability 1 for any bounded set $A \subset \mathbb{R}^+$. The measure $N(\omega, \cdot)$ depends on ω; it is thus a *random measure*. The intensity λ of the Poisson process determines the *average* value of the random measure $N(\cdot, A)$, that is

$$E[N(\cdot, A)] = \lambda |A|,$$

where $|A|$ is the Lebesgue measure of A.
$N(\omega, \cdot)$ is called a *Poisson random measure* associated with the Poisson process $N(t)$. The Poisson process $N(t)$ may be expressed in terms of the random measure N in the following way:

$$N(\omega, t) = N(\omega, [0, t]) = \int_{[0,t]} N(\omega, ds).$$

Conversely, the Poisson random measure N can also be viewed as the "derivative" of a Poisson process. Recall that each trajectory $t \mapsto N(\omega, t)$ of a Poisson process is an increasing step function. Hence its derivative (in the sense of distributions) is a positive measure on $\sigma-$algebra of Borel sets of $\mathbb{R}^+$. In fact, it is simply the superposition of Dirac masses located at the jump times:

$$\frac{d}{dt} N(\omega, t) = \sum_{n \geq 1} \delta_{T_n(\omega)}(\cdot) =: N(\omega, \cdot),$$

hence, for any predictable process $f(\omega, s)$, the stochastic integral with respect to the Poisson random measure N admits, for any $t \in \mathbb{R}^+$, the form

$$\int_0^t f(\cdot,s)N(\cdot,ds) = \sum_{n\geq 1} f(T_n)1_{\{T_n(\omega)\leq t\}}(\cdot) = \sum_{n=1}^{N(\cdot,t)} f(T_n),$$

or in a more compact form

$$\int\limits_0^t f(s)dN(s) = \sum_{n=1}^{N(t)} f(T_n). \tag{3}$$

We now assume that the T_n's correspond to the jump times of a Poisson process $N(t)$ and that Y_n is a sequence of indentically distributed random variables with values in $(-1,\infty)$. Let $S(t)$ be a predictable process. At time T_n the jump of the dynamics of $S(t)$ is given by

$$S(T_n) - S(T_n-) = S(T_n-)Y_n, \tag{4}$$

which, by the assumption $Y_n > -1$, leads always to positive values of the prices.

If $f(S,t)$ is a $C^{\{2,1\}}$-function (this means that f is C^2 in the first variable and C^1 in the second variable), then it follows from (3) that

$$\int_0^t [f(S(s-)(1+Y_s),s) - f(S(s-),s)]dN(s) = \sum_{n=1}^{N(t)} [f(S(T_n),T_n) - f(S(T_n-),T_n)] \tag{5}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. An application of equation (5) to the function $f(S,t) = S$ for $S \geq 0$ yields

$$\int_0^t [S(s-)(1+Y_s) - S(s-)]dN(s) = \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)]$$

or

$$\int_0^t S(s-)Y_s dN(s) = \sum_{n=1}^{N(t)} [S(T_n) - S(T_n-)]. \tag{6}$$

It then follow from equations (4) and (6) that

$$\int_0^t S(s-)Y_s dN(s) = \sum_{n=1}^{N(t)} S(T_n-)Y_n. \tag{7}$$

The following lemma is an Ito's formula for jump-diffusion process. Its proof can be found in [2, p. 275].

Lemma 1. *Let X be a diffusion process with jumps, defined as the sum of drift term, a Brownian stochastic integral and a compound Poisson process:*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dW(s) + \sum_{n=1}^{N(t)} \Delta X_n.$$

Here $b(t)$, $\sigma(t)$ are continuous nonanticipating processes with

$$E\left[\int_0^\tau \sigma^2(t)dt\right] < \infty,$$

and $\Delta X_n = X(T_n) - X(T_n-)$ *are the jump sizes. Then, for any* $C^{2,1}$ *function,* $f : \mathbb{R}\times[0,T]\to\mathbb{R}$*, the process* $Y(t) = f(X(t),t)$ *can be represented as:*

$$\begin{aligned} f(X(t),t) - f(X(0),0) = & \int_0^t \left[\frac{\partial f}{\partial x}(X(s),s)b(s) + \frac{\partial f}{\partial s}(X(s),s)\right] ds \\ & + \frac{1}{2}\int_0^t \sigma^2(s)\frac{\partial^2 f}{\partial x^2}(X(s),s)ds + \int_0^t \frac{\partial f}{\partial x}(X(s),s)\sigma(s)dW(s) \\ & + \sum_{n=1}^{N(t)} \left[f(X(T_n),T_n) + f(X(T_n-),T_n)\right]. \end{aligned}$$

3. Stock Price Model with Jumps

Let us consider a probability space $(\Omega, \boldsymbol{\mathcal{F}}, \mathbb{P})$ on which we define a standard Brownian motion $(W(t), t \geq 0)$, a Poisson process $(N(t), t \geq 0)$ with intensity λ and a sequence $(Y_n, n \geq 1)$ of independent, identically distributed random variables taking values in $(-1, +\infty)$. We will assume that the σ-algebras generated respectively by $(W(t), t \geq 0)$, $(N(t), t \geq 0)$ and $(Y_n, n \geq 1)$ are independent.

The objective of this section is to model a financial market in which there is one riskless asset (with price $S^0(t) = e^{\mu t}$, at time t) and one risky asset whose price jumps at the proportions $Y_1, \ldots, Y_n, \ldots$, at some times $T_1, \ldots, T_n, \ldots$ and which, between any two jumps, follows the Black-Scholes model. Moreover, we will assume that the T_n's correspond to the jump times of a Poisson process.

The dynamics of $S(t)$, the price of the risky asset at time t, can now be described in the following manner. The process $(S(t), t \geq 0)$ is an adapted, right-continuous process such that on the time intervals $[T_n, T_{n+1})$,

$$dS(t) = S(t)(\mu dt + \sigma dW(t)), 0 \leq t \leq T \tag{8}$$

while at $t = T_n$, the jump of $S(t)$ is given by

$$\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n.$$

Thus

$$S(T_n) = S(T_n-)(1+Y_n).$$

By using the standard Itô formula, the solution of (8) on the interval $[0, T_1)$ is

$$S(t) = S(0)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right).$$

Consequently, the left-hand limit at T_1 is given by

$$S(T_1-) = \lim_{u\to T_1} S(u) = S(0)\exp\left((\mu - \frac{\sigma^2}{2})T_1 + \sigma W(T_1)\right)$$

and

$$S(T_1) = S(0)(1+Y_1)\exp\left(\Big(\mu - \frac{\sigma^2}{2}\Big)T_1 + \sigma W(T_1)\right).$$

Then, for $t \in [T_1, T_2)$,

$$\begin{aligned} S(t) &= S(T_1)\exp(\mu - \frac{\sigma^2}{2})(t - T_1) + \sigma(W(t) - W(T_1)) \\ &= S(0)(1+Y_1)\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right). \end{aligned}$$

Repeating this scheme, we obtain

$$S(t) = S(0)\left[\prod_{N(t)} n = 1(1+Y_n)\right]\exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right) \tag{9}$$

with the convention $\prod_0 n = 1 = 1$. Using equation (3), $S(t)$ can be given in the following equivalent representations

$$\begin{aligned} S(t) &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \log\left(\prod_{n=1}^{N(t)}(1+Y_n)\right)\right] \\ &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{n=1}^{N(t)}\log(1+Y_n)\right] \\ &= S(0)\exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \int_0^t \log(1+Y_s)dN(s)\right], \end{aligned}$$

where Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation.

The process $(S(t), t \geq 0)$ in equation (9) is right-continuous, adapted and has only finitely many discontinuities on each interval $[0, t]$. We can also prove the following.

Theorem 1. *For all $t \geq 0$, $(S(t), t \geq 0)$ in equation (9) satisfies:*

$$\mathbb{P} \quad a.s. \quad S(t) = S(0) + \int_0^t S(s)(\mu ds + \sigma dW(s)) + \sum_{n=1}^{N(t)} S(T_n-)Y_n \tag{10}$$

or, in differential form

$$\mathbb{P} \quad a.s. \quad dS(t) = S(t)(\mu dt + \sigma dW(t)) + S(t-)Y_t dN(t). \tag{11}$$

Proof. Let $\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n$. Then (10) can be written in the following form:

$$\mathbb{P} \quad a.s. \quad S(t) = S(0) + \int_0^t S(s)\left(\mu ds + \sigma dW(s)\right) + \sum^{N(t)} n = 1\Delta S_n, \tag{12}$$

We choose the function $f(x,s) = log\ x$. Direct calculation shows that

$$f_x = \frac{1}{x}, f_{xx} = -\frac{1}{x^2} \quad \text{and} \quad f_s = 0$$

We note that $f(x,t)$ is a$C^{2,1}$ function if $x > 0$. Assume that $S(t)$ in (10) is nonnegative. Applying the Itô formula for jump-diffusion processes (see Lemma 1) to $f(x,t) = \log x$, we obtain

$$\log S(t) = \log S(0)(\mu - \frac{\sigma^2}{2})t + \sigma W(s) + \sum_{n=1}^{N(t)} \log(1 + Y_n).$$

Thus,

$$S(t) = S(0) \left[\prod_{n=1}^{N(t)} (1 + Y_n) \right] \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W(t) \right).$$

Hence, we obtain (9) as asserted.

4. A Fractional Stock Price Model with Jumps

We use the same setting probability spaces as in Sec. 3. The objective of this section is to construct an approximate model for a financial market in which there is one riskless asset (with price $S^0(t) = e^{\mu t}$, at time t) and one risky asset whose price jumps in the proportions $Y_1, \ldots, Y_n, \ldots$ at some random times $T_1, T_2, ..., T_n, ...$and which, between two jumps, follows the fractional Black-Scholes model.for a fractional process $B(t)$. These descriptions can be formalized on the intervals $[T_n, T_{n+1})$ by letting:

$$dS(t) = S(t)(\mu dt + \sigma dB(t)), 0 \leq t \leq T. \tag{13}$$

At $t = T_n$, the jump of $S(t)$ is given by

$$\Delta S_n = S(T_n) - S(T_n-) = S(T_n-)Y_n.$$

Now, we consider a *fractional Black-Scholes model with jumps* which is defined similarly to equation (11) by the following stochastic differential equation

$$\begin{aligned} dS(t) &= S(t)(\mu dt + \sigma dB(t)) + S(t-)Y_t dN(t), \\ S(t)|_{t=0} &= S(0). \end{aligned} \tag{14}$$

Here $B(t) = \int_0^t (t-s)^\alpha dW(s)$ where $0 < \alpha < \frac{1}{2}$.

The corresponding approximate model of (14) is defined for each $\varepsilon > 0$ by

$$\begin{aligned} dS_\varepsilon(t) &= S_\varepsilon(t)(\mu dt + \sigma dB_\varepsilon(t)) + S_\varepsilon(t-)Y_t dN(t), \\ S_\varepsilon(t)|_{t=0} &= S(0) \quad \text{(same intial conditon as in (14))}, \end{aligned} \tag{15}$$

where $B_\varepsilon(t) = \int_0^t (t-s+\varepsilon)^\alpha dW(s)$. One can prove that $B_\varepsilon(t)$ is a semimartingale and $B_\varepsilon(t)$ converges to $B(t)$ in $L^2(\Omega)$ when $\varepsilon \to 0$. This convergence is uniform with respect to $t \in [0, T]$ (see [10, Theorem 2.1]). We need the following Lemma considered as a consequence of the L^2-convergence of $B_\varepsilon(t)$ to $B(t)$.

Lemma 2. *$B_\varepsilon(t)$ converges to $B(t)$ in $L^p(\Omega)$ for any $p \geq 2$, uniformly with respect to $t \in [0, T]$.*

Proof. The proof of this Lemma is due to Nguyen Tien Dung [8].

Theorem 2. *Suppose that $S(0)$ is a random variable such that $E|S(0)|^{2+\delta}$ is finite for some $\delta > 0$. Then the solution of (15) is given by:*

$$S_\varepsilon(t) = S(0) \exp\left(-\frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \sigma\varepsilon^\alpha W(t) + \int_0^t H_\varepsilon(s)ds + \int_0^t \log(1 + Y_s)dN(s)\right),$$

where $0 < \alpha < \frac{1}{2}$, and

$$H_\varepsilon(t) = \mu + \alpha\sigma \int_0^t (t - s + \varepsilon)^{\alpha-1} dW(s).$$

Furthermore, the stochastic process $S_*(t)$ defined by

$$S_*(t) = S(0) \exp\left(\mu t + \sigma B(t) + \int_0^t \log(1 + Y_s)dN(s)\right)$$

is the limit in $L^2(\Omega)$ of $S_\varepsilon(t)$ as $\varepsilon \to 0$. This limit is uniform with respect to $t \in [0, T]$.

Proof. Letting $\varphi_\varepsilon(t) = \int_0^t (t - s + \varepsilon)^{\alpha-1} dW(s)$, and substituting $dB_\varepsilon(t) = \alpha\varphi_\varepsilon(t)dt + \varepsilon^\alpha dW(t)$ into equation (eqn15), we obtain

$$dS_\varepsilon(t) = [\mu + \alpha\sigma\varphi_\varepsilon(t)]S_\varepsilon(t)dt + \sigma\varepsilon^\alpha S_\varepsilon(t)dW(t) + S_\varepsilon(t-)Y_t dN(t), \qquad (16)$$

or,

$$\begin{aligned} \frac{dS_\varepsilon(t)}{S_\varepsilon(t)} &= [\mu + \alpha\sigma\varphi_\varepsilon(t)]dt + \sigma\varepsilon^\alpha dW(t) + \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right) Y_t dN(t) \\ &= H_\varepsilon(t)dt + \sigma\varepsilon^\alpha dW(t) + \left(\frac{S_\varepsilon(t-)}{S_\varepsilon(t)}\right) Y_t dN(t) \qquad (17) \end{aligned}$$

where $H_\varepsilon(t) = \mu + \alpha\sigma\varphi_\varepsilon(t)$. Moreover, we can write equation (eqn16) into an integral form as

$$\int_0^t dS_\varepsilon(t) = \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \int_0^t S_\varepsilon(s-)Y_s dN(s).$$

Thus,

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s)S_\varepsilon(s)ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s)dW(s) + \int_0^t S_\varepsilon(s-)Y_s dN(s).$$

Using the formula (7), $S_\varepsilon(t)$ can be given in the following equivalent representations

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s) S_\varepsilon(s) ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s) dW(s) + \sum_{n=1}^{N(t)} S_\varepsilon(T_n-) Y_n. \tag{18}$$

Since $\Delta S_\varepsilon(T_n) = S_\varepsilon(T_n) - S_\varepsilon(T_n-) = S_\varepsilon(T_n-)Y_n$ then equation (18) becomes

$$S_\varepsilon(t) = S(0) + \int_0^t H_\varepsilon(s) S_\varepsilon(s) ds + \int_0^t \sigma\varepsilon^\alpha S_\varepsilon(s) dW(s) + \sum_{n=1}^{N(t)} \Delta S_\varepsilon(T_n).$$

Choosing the function $f(x, s) = \log x$ for $x = S_\varepsilon(t) > 0$, direct calculation shows that

$$f_x = \frac{1}{x}, f_{xx} = -\frac{1}{x^2} \quad \text{and} \quad f_s = 0$$

An application of the Itô formula for jump-diffusion processes (see Lemma 1) gives:

$$\begin{aligned}
\log S_\varepsilon(t) &= \log S(0) + \int_0^t \left(0 + \left(\frac{1}{S_\varepsilon(s)} \right) \cdot (H_\varepsilon(s) S_\varepsilon(s)) \right) ds \\
&+ \frac{1}{2} \int_0^t (\sigma\varepsilon^\alpha)^2 S_\varepsilon^2(s) \left(-\frac{1}{S_\varepsilon(s)} \right)^2 ds \\
&+ \int_0^t \left(\frac{1}{S_\varepsilon(s)} \right) (\sigma\varepsilon^\alpha) S_\varepsilon(s) dW(s) \\
&+ \sum_{n=1}^{N(t)} [\log(S_\varepsilon(T_n-) + \Delta S_\varepsilon(T_n)) - \log(S_\varepsilon(T_n-))] \\
&= \log S(0) + \int_0^t H_\varepsilon(s) ds - \frac{1}{2} \int_0^t (\sigma\varepsilon^\alpha)^2 ds + \int_0^t \sigma\varepsilon^\alpha dW(s) \\
&+ \sum_{n=1}^{N(t)} \left[\log \left(\frac{S_\varepsilon(T_n-)(1 + Y_n)}{S_\varepsilon(T_n-)} \right) \right] \\
&= \log S(0) + \int_0^t (H_\varepsilon(s) ds + \sigma\varepsilon^\alpha dW(s)) - \frac{1}{2} \int_0^t (\sigma\varepsilon^\alpha)^2 ds \qquad (19) \\
&+ \sum_{n=1}^{N(t)} \log(1 + Y_n)
\end{aligned}$$

Using formulae (7) and (17), equation (19) can be given in the following equivalent representations

$$
\begin{aligned}
\log S_\varepsilon(t) &= \log S(0) + \int_0^t (H_\varepsilon(s)ds + \sigma\varepsilon^\alpha dW(s)) - \frac{1}{2}\int_0^t (\sigma\varepsilon^\alpha)^2 ds \\
&+ \int_0^t \log(1+Y_n)dN(s) \\
&= \log S(0) + \left(\int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} - \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s)\right) - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t \\
&+ \int_0^t \log(1+Y_n)dN(s) \\
&= \log S(0) + \int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} - \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t + \int_0^t \log(1+Y_n)dN(s) \\
&- \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s).
\end{aligned}
$$

Here Y_t is obtained from Y_n by a piecewise constant and left continuous time interpolation. Thus

$$
\int_0^t \frac{dS_\varepsilon(s)}{S_\varepsilon(s)} = \log\frac{S_\varepsilon(t)}{S(0)} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t - \int_0^t \log(1+Y_n)dN(s) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s). \tag{20}
$$

Equating (20) and (17), we get

$$
\begin{aligned}
&\log\frac{S_\varepsilon(t)}{S(0)} + \frac{1}{2}\sigma^2\varepsilon^{2\alpha}t - \int_0^t \log(1+Y_n)dN(s) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s) \\
&= \int_0^t H_\varepsilon(s)ds + \sigma\varepsilon^\alpha W(t) + \int_0^t \left(\frac{S_\varepsilon(s-)}{S_\varepsilon(s)}\right) Y_s dN(s).
\end{aligned}
$$

Hence, the solution of (15) is

$$
S_\varepsilon(t) = S(0)\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma\varepsilon^\alpha W(t) + \int_0^t H_\varepsilon(s)ds + \int_0^t \log(1+Y_n)dN(s)\right). \tag{21}
$$

We note that,

$$
\int_0^t H_\varepsilon(s)ds = \mu + \alpha\sigma\int_0^t \varphi_\varepsilon(s)ds.
$$

By application of the stochastic Theorem of Fubini, we get

$$
\int_0^t \varphi_\varepsilon(s)ds = \frac{1}{\alpha}\left(B_\varepsilon(t) - \varepsilon^\alpha W(t)\right).
$$

Therefore

$$
\int_0^t H_\varepsilon(s)ds = \mu t + \sigma B_\varepsilon(t) - \sigma\varepsilon^\alpha W(t).
$$

Substituting the value of $\int_0^t H_\varepsilon(s)ds$ into equation (21), we get

$$S_\varepsilon(t) = S(0)\exp\left(\mu t - \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_n)dN(s)\right).$$

We note that $\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t \to 0$ as $\varepsilon \to 0$ and $B_\varepsilon(t)$ converges uniformly to $B(t)$ in $L^2(\Omega)$ when $\varepsilon \to 0$. This motivates us to consider the process $S_*(t)$ defined by

$$S_*(t) = S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right).$$

We try to show that $S_*(t)$ is the limit of $S_\varepsilon(t)$ in $L^2(\Omega)$ as $\varepsilon \to 0$. We observe that

$$\begin{aligned}
S_\varepsilon(t) - S_*(t) =& S(0)\exp\left(\mu t - \frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma B_\varepsilon(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\
&- S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\
&= S(0)\exp\left(\mu t + \sigma B(t) + \int_0^t \log(1+Y_n)dN(s)\right) \\
&\left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] \\
&= S(0)\exp\left(\mu t + \sigma B(t)\right) \cdot \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) \\
&\left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right].
\end{aligned}$$

Put $p = 1 + \frac{\delta}{2}$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. It follows from Holder's inequality that

$$\begin{aligned}
||S_\varepsilon(t) - S_*(t)||_2 \leq& ||S(0)||_{2p}|| \exp\left(\mu t + \sigma B(t)\right) \cdot \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) \times \\
&\left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] ||_{2q} \\
&\leq ||S(0)||_{2+\delta}|| \exp\left(\mu t + \sigma B(t)\right) \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{4q} \times \\
&|| \left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2 t + \sigma(B_\varepsilon(t) - B(t))\right) - 1\right] ||_{4q}
\end{aligned} \tag{22}$$

In order to calculate the norm $||S_\varepsilon(t) - S_*(t)||_2$, we firstly note that

$$\begin{aligned}
&|| \exp\left(\mu t + \sigma B(t)\right) \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{4q} \\
&\leq || \exp\left(\mu t + \sigma B(t)\right) ||_{8q} || \exp\left(\int_0^t \log(1+Y_n)dN(s)\right) ||_{8q} < \infty.
\end{aligned} \tag{23}$$

To see this we note that, for each t, B_t is a Gaussian random variable with zero mean and variance γ_t^2 for some real numbers γ_t. Then

$$||\exp\left(\mu t+\sigma B(t)\right)||_{8q}=\exp(\mu t)[Ee^{8q\sigma B(t)}]^{\frac{1}{8q}}=\exp(\mu t)e^{4q\sigma^2\gamma^2(t)}<\infty.$$

Moreover

$$\left\|\exp\left(\int_0^t\log(1+Y_n)dN(s)\right)\right\|_{8q}=\left\|\exp\left(\sum_{n=1}^{N(t)}\log(1+Y_n)\right)\right\|_{8q}=\left\|\sum_{n=1}^{N(t)}(1+Y_n)\right\|_{8q}\leq K,$$

where K is a constant. This is due to the fact that there is a finite number of jumps in the finite interval $[0,T]$.
Finally, we compute the last term on the right hand side of (22). It follows from the relation $e^A-1=A+o(A)$ that we have

$$\begin{aligned}
&\left\|\left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t))\right)-1\right]\right\|_{4q}\\
&\leq\left\|-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t))\right\|_{4q}+\left\|o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t)))\right\|_{4q}\\
&\leq\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma\left\|B_\varepsilon(t)-B(t)\right\|_{4q}+||o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t)))||_{4q}
\end{aligned}$$

By application of Lemma 2, we have $||B_\varepsilon(t)-B(t)||_{4q}\to 0$ as $\epsilon\to 0$ (uniformly on $t\in[0,T]$). Hence

$$\begin{aligned}
\left\|\left[\exp\left(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t))\right)-1\right]\right\|_{4q}&\leq\frac{1}{2}(\sigma\varepsilon^\alpha)^2T+\sigma||B_\varepsilon(t)-B(t)||_{4q}+\\
&\left\|o(-\frac{1}{2}(\sigma\varepsilon^\alpha)^2t+\sigma(B_\varepsilon(t)-B(t)))\right\|_{4q}
\end{aligned}$$

The right hand side of the above inequality does not depend on t and approaches zero when $\varepsilon\to 0$. Therefore, one can see from (22) and (23), that $S_\varepsilon(t)\to S_*(t)$ in $L^2(\Omega)$ as $\varepsilon\to 0$ and the convergence is uniform with respect to t.

5. Simulation Examples

Let us consider the Thai stock market. Figure 1 shows the daily prices of a data set consisting of 150 open -prices of the Thai Petrochemical Industry (TPI) between June 9, 2004 and January 7, 2005. The empirical data for these stock prices were obtained from http://finance.yahoo.com.

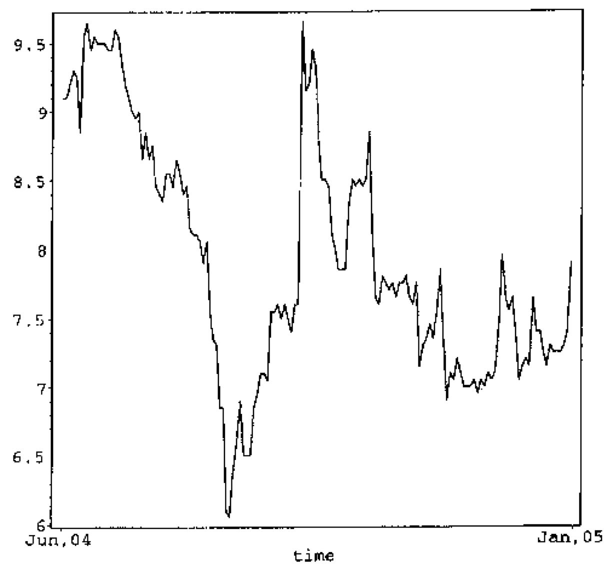

Fig. 1. Price behavior of *TPI, between June 4, 2004 and January 7, 2005*

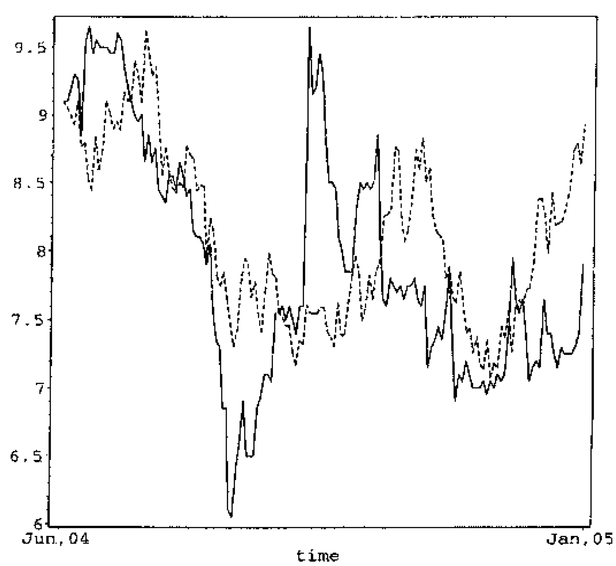

Fig. 2. Price behavior of *TPI, between June 4, 2004 and January 7, 2005,* compared with a scenario simulated from a Black-Scholes model with jumps (solid line:= empirical data, dashed line:= simulated by $S(t) = S(0)\exp((\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{n=1}^{N(t)}(1+Y_n)))$, *ARPE(2) = 23.69%, and variance = 0.02656)*

Figure 2 shows the empirical data of TPI open-price as compared to the price that was simulated by a Black-Scholes pricing model with jump. In the simulation process, we use the algorithm that appeared the paper of Cyganowski, Grunce and Kloeden [3]. The simulated model is $S(t) = S(0)\exp((\mu - \frac{\sigma^2}{2})t + \sigma W(t) + \sum_{n=1}^{N(t)}(1+Y_n)$. The model parameters $\mu = -0.0000725, \sigma = 0.3025$ and parameter for jumps as $\mu_j = 0.00007624$, $\sigma_j = 0.0003679$, $\lambda = 55.46$, $\gamma = 1$ are fixed. For comparative purposes, we compute the Average Relative Percentage Error(ARPE). By definition, ARPE= $(1/N)\sum_{k=1}^{N}\frac{|X_k - Y_k|}{X_k}.100$, where N is the number of price, $X = (X_k)_{k\geq 1}$ is the market prices and $Y = (Y_k)_{k\geq 1}$ is the model prices. We worked out 500 trails and computed ARPE. We denote the ARPE of Figure 2 and and Figure 3 by ARPE(2) and ARPE(3) respectively.

Figure 3 shows the empirical data of TPI open-price as compared to the price that was simulated by a fractional Black-Scholes pricing model with jumps. The simulated model is $S_{\in}(t) = S(0)\exp((\mu - \frac{1}{2}((\sigma\varepsilon^{\alpha})^2)t + \sigma B_{\in}(t) + \sum_{n=1}^{N(t)}(1+Y_n))$. The value of μ, σ and the parameters for jumps are the same as in Figure 2. For the remaining data, we choose $H = 0.50001$, $\varepsilon = 0.000001$.

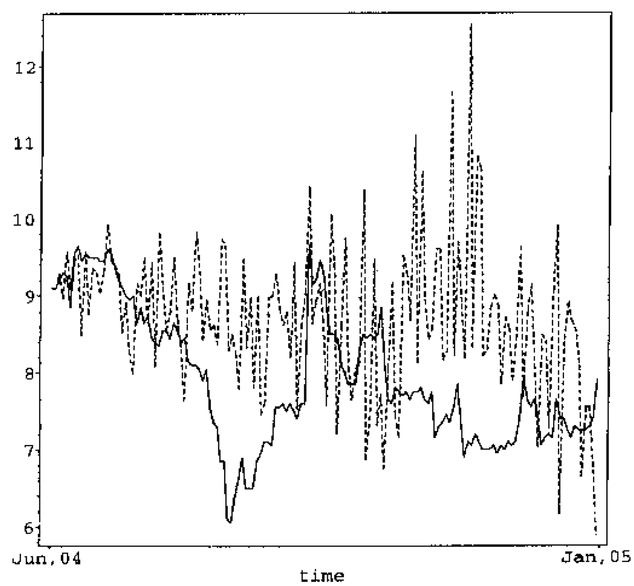

Fig. 3. Price behavior of *TPI, between June 4, 2004 and January 7, 2005,* compared with a scenario simulated from a fractional Black Scholes model with jumps (solid line := empirical data, dashed line := simulated by $S_{\varepsilon}(t)=S(0)\exp((\mu-\frac{1}{2}((\sigma\varepsilon^{\alpha})^2)t+\sigma B_{\in}(t)+\sum_{n=1}^{N(t)}(1+Y_n))$. ARPE(3) = 19.64%, and variance = 0.01546)

By comparing ARPE and varience of Figure 2 and 3, one can see that in case of TPI, the sample path from a fractional Black-Scholes pricing model with jumps gives a better fit with the data than Black-Scholes pricing model with jumps.

References

1. E. Alos, O. Mazet, and D. Nualart, Stochastic calculus with respect to fractional Brownian motion with Hurst parameter less than $\frac{1}{2}$, *Stochastic Processes and their Applications* **86** (2000) 121–139.
2. R. Cont and P. Tankov, *Financial Modelling with Jump Processes,* Chapman & Hall/CRC 2004.
3. S. Cyganowski, L. Grunce, and P. E. Kloeden, MAPLE for jump-diffusion stochastic differential equations in finance, Available at http://www.uni-bayreuth.de/ departments/ math/~lgruene/papers, 2002.
4. T. E. Duncan, Y. Z. Hu, and B. Parsik-Duncan, Stochastic calculus for fractional Brownian motion I, Theory, *SIAM J. Control and Optim.* **38** (2000) 582–615.
5. R. J. Elliot and J. Van der Hoek, A general white noise theory and applications to finance, *Math. Finance* **13** (2003) 301–330.
6. Y. Hu and B. Oksendal, Fractional white noise calculus and applications to finance, *Infin. Dimens. Anal. Quantum probab. Relat.* **6** (2003) 1–32.
7. D. Lamberton and B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance,* Chapman & Hall, 1996.
8. Nguyen Tien Dung, A class of fractional stochastic equation, Institute of mathematics, Vietnam Academy of Science and Technology, 2007.
9. B. Oksendal, *Fractional Brownian Motion in Finance,* Preprint Department of Math, University of Oslo **28** (2003) 1–35.
10. T. H. Thao, An approximate Approach to fractional analysis for Finance, *Nonlinear Analysis: Real world Applications* **7** (2006) 124–132.