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Option pricing for a stochastic volatility Lévy model with stochastic interest rates

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ABSTRACT

An alternative option pricing model under a forward measure is proposed, in which asset prices follow a stochastic volatility Lévy model with stochastic interest rate. The stochastic interest rate is driven by the Hull–White process. By using an approximate method, we find a formulation for the European option in term of the characteristic function of the tail probabilities.

1. Introduction

Let (Ω, F, P) be a probability space. A stochastic process L_t is a Lévy process if it has independent and stationary increments and has a stochastically continuous sample path, i.e. for any $\varepsilon > 0$, $\lim_{h\downarrow 0} P(|L_{t+h} - L_t| > \varepsilon) \to 0$. The simplest possible Lévy processes are the standard Brownian motion W_t, Poisson process N_t, and compound Poisson process $\sum_{i=1}^{N_t} Y_i$ where Y_i are i.i.d. random variables. Of course, we can build a new Lévy process from known ones by using the technique of linear transformation. For example, the jump diffusion process $\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$, where μ, σ are constants, is a Lévy process which comes from a linear transformation of two independent Lévy processes, i.e. a Brownian motion with drift and a compound Poisson process.

Assume that a risk-neutral probability measure Q exists and all processes in Sections 1 and 2 will be considered under this risk-neutral measure.

In the Black–Scholes model, the price of a risky asset S_t under a risk-neutral measure Q and with non dividend payment follows

$$S_t = S_0 \exp(\tilde{L}_t) = S_0 \exp\left(rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)\right), \tag{1.1}$$

where $r \in \Re$ is a risk-free interest rate, $\sigma \in \Re$ is a volatility coefficient of the stock price.

Instead of modeling the log returns $\tilde{L}_t = rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right)$ with a normal distribution, we now replace it with a more sophisticated process L_t which is a Lévy process of the form

$$L_t = rt + \left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) + J_t, \tag{1.2}$$

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where J_t denotes a pure Lévy jump component, (i.e. a Lévy process with no Brownian motion part). We assume that the processes W_t and J_t are independent.

To incorporate the volatility effect to the model Eq. (1.2), we follow the technique of Carr and Wu (2004) by subordinating a standard Brownian motion component $\sigma W_t - \frac{1}{2}\sigma^2 t$ and a pure jump Lévy process J_t by the time integral of a mean reverting Cox–Ingersoll–Ross (CIR) process

$$T_t = \int_0^t v_s ds,$$

where v_t follows the CIR process

$$dv_t = \gamma(1 - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v. \tag{1.3}$$

Here W_t^v is a standard Brownian motion which corresponds to the process v_t. The constant $\gamma \in \mathbb{R}$ is the rate at which the process v_t reverts toward its long term mean and $\sigma_v > 0$ is the volatility coefficient of the process v_t.

Hence, the model (1.2) has been changed to

$$L_t = rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + J_{T_t} \tag{1.4}$$

and this new process is called a stochastic volatility Lévy process. One can interpret T_t as the stochastic clock process with activity rate process v_t. By replacing $\tilde{L}_t$ in (1.1) with L_t, we obtain a model of an underlying asset under the risk-neutral measure Q with stochastic volatility as follows:

$$S_t = S_0 \exp(L_t) = S_0 \exp\left(rt + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + J_{T_t}\right). \tag{1.5}$$

In this paper, we shall consider the problem of finding a formula for European call options based on the underlying asset model (1.5) for which the constant interest rates r is replaced by the stochastic interest rates r_t, i.e. the model under our consideration is given by

$$S_t = S_0 \exp\left(r_t t + \left(\sigma W_{T_t} - \frac{1}{2}\sigma^2 T_t\right) + J_{T_t}\right). \tag{1.6}$$

Here, we assume that r_t follows the Hull–White process

$$dr_t = (\alpha(t) - \beta r_t)dt + \sigma_r dW_t^r, \tag{1.7}$$

W_t^r is a standard Brownian motion with respect to the process r_t, and $dW_t^r dW_t^=0$. The constant $\beta \in \mathbb{R}$ is the rate at which the interest rate reverts toward its long term mean, $\sigma_r > 0$ is the volatility coefficient of the interest rate process (1.7), $\alpha(t)$ is a deterministic function, and is well defined in a time interval $[0, T]$. We also assume that the interest rate process r_t and the activity rate process v_t are independent.

The problem of option pricing under stochastic interest rates has been investigated for a long time. Kim (2001) constructed the option pricing formula based on Black–Scholes model under several stochastic interest rate processes, i.e., Vasicek, CIR, Ho–Lee type. He found that by incorporating stochastic interest rates into the Black–Scholes model, for a short maturity option, does not contribute to improvement in the performance of the original Black–Scholes' pricing formula. Brigo and Mercuiro (2001, p. 883) mention that the stochastic feature of interest rates has a stronger impact on the option price when pricing for a long maturity option. Carr and Wu (2004) continue this study by giving the option pricing formula based on a time-changed Levy process model. But they still use constant interest rates in the model.

In this paper, we give an analysis on the option pricing model based on a time-changed Levy process with stochastic interest rates.

The rest of the paper is organized as follows. The dynamics under the forward measure is described in Section 2. The option pricing formula is given in Section 3. Finally, the close form solution for a European call option in terms of the characteristic function is given in Section 4.

2. The dynamics under the forward measure

We begin by giving a brief review of the definition of a correlated Brownian motion and some of its properties (see Brummelhuis (2009, p. 70)). Recalling that a *standard Brownian motion in* R^n is a stochastic process $(\vec{Z}_t)_{t\geq 0}$ whose value at time t is simply a vector of n independent Brownian motions at t:

$$\vec{Z}_t = (Z_{1,t}, \ldots, Z_{n,t}).$$

We use Z instead of W, since we would like to reserve the latter for the more general case of correlated Brownian motion, which will be defined as follows:

Let $\rho = (\rho_{ij})_{1\le i,j\le n}$ be a (constant) positive symmetric matrix satisfying $\rho_{ii} = 1$ and $-1 \le \rho_{ij} \le 1$. By Cholesky's decomposition theorem, one can find an upper triangular $n \times n$ matrix $H = (h_{ij})$ such that $\rho = HH^t$, where H^t is the transpose of the matrix H. Let $\vec{Z}_t = (Z_{1,t}, \dots, Z_{n,t})$ be a standard Brownian motion as introduced above, we define a new vector-valued process $\vec{W}_t = (W_{1,t}, \dots, W_{n,t})$ by $\vec{W}_t = H\vec{Z}_t$ or, in terms of components,

$$W_{i,t} = \sum_{j=1}^{n} h_{ij} Z_{j,t}, \quad i = 1, \dots, n.$$

The process $(\vec{W}_t)_{t\ge 0}$ is called a *correlated Brownian motion* with a (constant) correlation matrix ρ. Each component-process $(W_{i,t})_{t\ge 0}$ is itself a standard Brownian motion. Note that if $\rho = Id$ (the identity matrix) then $\vec{W}_t$ is a standard Brownian motion. For example, if we let a symmetric matrix

$$\rho = \begin{bmatrix} 1 & \rho_v & 0 \\ \rho_v & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.1}$$

Then ρ has a *Cholesky decomposition* of the form $\rho = HH^T$ where H is an upper triangular matrix of the form

$$H = \begin{bmatrix} \sqrt{1-\rho_v^2} & \rho_v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\vec{Z}_t = (Z_t, Z_t^r, Z_t^v)$ be three independent Brownian motions, then $\vec{W}_t = (W_t, W_t^r, W_t^v)$ defined by $\vec{W}_t = H\vec{Z}_t$, or in terms of components,

$$W_t = \left(\sqrt{1-\rho_v^2}\right) Z_t + \rho_v Z_t^v, \qquad W_t^v = Z_t^v, \qquad W_t^r = Z_t^r, \tag{2.2}$$

is a correlated Brownian motion with correlation matrix ρ as given in Eq. (2.1).

Now let us turn to our problem. Note that, by Ito's lemma, the model (1.6) has the dynamic given by

$$\begin{aligned} dS_t &= S_{t-} r_t dt + \sigma S_{t-} dW_{T_t} + S_{t-} dJ^*_{T_t}, \\ dr_t &= (\alpha(t) - \beta r_t) dt + \sigma_r dW_t^r, \\ dv_t &= \gamma(1 - v_t) dt + \sigma_v \sqrt{v_t} dW_t^v, \end{aligned} \tag{2.3}$$

where $dJ^*_{T_t} = dJ_{T_t} + (e^{\Delta J_{T_t}} - 1 - \Delta J_{T_t})$, $dW_t dW_t^r = dW_t^r dW_t^v = 0$, and $dW_t dW_t^v = \rho_v dt$.

We can re-write the system (2.3) in terms of three independent Brownian motions (Z_t, Z_t^v, Z_t^r) as follows:

$$dS_t = S_{t-} r_t dt + \sigma S_{t-} \left(\rho_v dZ^v_{T_t} + \sqrt{1-\rho_v^2} dZ_{T_t} \right) + S_{t-} dJ^*_{T_t}, \tag{2.4}$$

$$dr_t = (\alpha(t) - \beta r_t) dt + \sigma_r dZ_t^r, \tag{2.5}$$

$$dv_t = \gamma(1 - v_t) dt + \sigma_v \sqrt{v_t} dZ_t^v. \tag{2.6}$$

This decomposition makes it easier to perform a measure transformation. In fact, for any fixed maturity T, let us denote by Q^T the T-forward measure, i.e. the probability measure that is defined by the Radon–Nikodym derivative,

$$\frac{dQ^T}{dQ} = \frac{\exp\left(-\int_0^T r_u du\right)}{P(0,T)}. \tag{2.7}$$

Here, $P(t,T)$ is the price at time t of a zero-coupon bond with maturity T and is defined as

$$P(t,T) = E_Q\left[e^{-\int_t^T r_s ds} | F_t \right]. \tag{2.8}$$

We denote $f(0,t)$ to be the market instantaneous forward rate at time 0 for the maturity time $t \ge 0$ and it is defined by

$$f(0,t) := -\frac{\partial}{\partial t} \ln P(0,t), \quad 0 \le t \le T. \tag{2.9}$$

Poulsen (2005) gave a relation between the coefficients of Eq. (2.5) and the forward rate $f(0,t)$ as follows:

$$\alpha(t) = \frac{\partial f(0,t)}{\partial t} + \beta f(0,t) + \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta t}\right). \tag{2.10}$$

Lemma 1. *The process r_t which satisfies the dynamic in (2.5) can be written in the form*

$$r_t = x_t + \varphi(t), \quad 0 \le t \le T, \tag{2.11}$$

where the process x_t satisfies

$$dx_t = -\beta x_t dt + \sigma_r dZ_t^r, \qquad x_0 = 0. \tag{2.12}$$

Moreover, the function φ is deterministic, well defined in the time interval $[0, T]$, and satisfies

$$\varphi(t) = f(0, t) + \frac{\sigma_r^2}{2\beta^2}\left(1 - e^{-\beta t}\right)^2. \tag{2.13}$$

In particular, $\varphi(0) = r_0$.

Proof. To find a solution of SDE (2.5), we let $g(t, r) = e^{\beta t} r$. By using Ito's Lemma, we have

$$dg = de^{\beta t} r_t = \alpha(t) e^{\beta t} dt + e^{\beta t} \sigma_r dZ_t^r. \tag{2.14}$$

Integrating on both sides of the above equation from 0 to t, we obtain

$$r_t = r_0 e^{-\beta t} + \int_0^t \alpha(u) e^{\beta(u-t)} du + \int_0^t e^{\beta(u-t)} \sigma_r dZ_u^r. \tag{2.15}$$

Substituting the value of $\alpha(t)$ from Eq. (2.10) into (2.15), we have

$$r_t = r_0 e^{-\beta t} + \int_0^t \left(\frac{\partial f(0, u)}{\partial u} + \beta f(0, u) + \frac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta u}\right)\right) e^{-\beta(t-u)} du + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{2.16}$$

Applying integration by parts formula to Eq. (2.16) and after simplifying, we obtain

$$r_t = r_0 e^{-\beta t} + f(0, t) - f(0, 0) e^{-\beta t} + \frac{\sigma_r^2}{2\beta^2}\left(e^{-\beta t} - 1\right)^2 + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{2.17}$$

By using the definition of φ_t from Eq. (2.13), we can write Eq. (2.17) into a compact form as follows:

$$r_t = r_0 e^{-\beta t} - f(0, 0) e^{-\beta t} + \varphi(t) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r = \varphi(t) + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r, \tag{2.18}$$

because of $f(0, 0) = r_0$, see Andrew (2004, p. 89).

Note that the solution of Eq. (2.12) is

$$x_t = x_0 e^{-\beta t} + \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r = \sigma_r \int_0^t e^{-\beta(t-u)} dZ_u^r. \tag{2.19}$$

Hence, $r_t = \varphi(t) + x_t$, $0 \le t \le T$. The proof is now complete. □

Now we are ready to calculate the Radon–Nikodym derivative as it appears in Eq. (2.7). By virtue of Lemma 1, $r_t = \varphi_t + x_t$. Substituting r_t and $P(0, T) = \exp\left(-\int_0^T f(0, u) du\right)$ into Eq. (2.7), one gets

$$\frac{dQ^T}{dQ} = \exp\left(-\int_0^T x_u du - \frac{\sigma_r^2}{2\beta^2}\int_0^T \left(1 - e^{-\beta(T-u)}\right)^2 du\right). \tag{2.20}$$

Stochastic integration by parts implies

$$\int_0^T x_u du = T x_T - \int_0^T u dx_u = \int_0^T (T - u)\, dx_u. \tag{2.21}$$

By substituting the expression for dx_u from Eq. (2.12), we have

$$\int_0^T (T - u)\, dx_u = -\beta \int_0^T (T - u) x_u du + \sigma_r \int_0^T (T - u) dZ_u^r. \tag{2.22}$$

Moreover, by substituting the expression for x_u from Eq. (2.19) into the right hand side of Eq. (2.22), one gets

$$-\beta \int_0^T (T - u) x_u du = -\beta \sigma_r \int_0^T \left((T - u) \int_0^u e^{-\beta(u-s)} dZ_s^r\right) du. \tag{2.23}$$

Using integration by parts, we have

$$-\beta\sigma_r \int_0^T \left((T-u) \int_0^u e^{-\beta(u-s)} dZ_u^r \right) du = -\frac{\sigma_r}{\beta} \left[\int_0^T \left(e^{-\beta(T-u)} - 1 \right) dZ_u^r \right] - \sigma_r \int_0^T (T-u) dZ_u^r. \tag{2.24}$$

Substituting Eq. (2.24) into (2.22) and Eq. (2.21) becomes

$$\int_0^T x_u du = -\frac{\sigma_r}{\beta} \left[\int_0^T \left(e^{-\beta(T-u)} - 1 \right) dZ_u^r \right]. \tag{2.25}$$

Substituting Eq. (2.25) into (2.20), we obtain

$$\frac{dQ^T}{dQ} = \exp\left(-\frac{\sigma_r}{\beta} \int_0^T \left(1 - e^{-\beta(T-u)} \right) dZ_u^r - \frac{\sigma_r^2}{2\beta^2} \int_0^T \left(1 - e^{-\beta(T-u)} \right)^2 du \right). \tag{2.26}$$

Hence, by Girsanov's theorem, the three processes Z_t^{rT}, Z_t^{vT} and Z_t^T defined by

$$dZ_t^{rT} = dZ_t^r + \frac{\sigma_r}{\beta} \left(1 - e^{-\beta(T-t)} \right) dt, \qquad dZ_t^{vT} = dZ_t^v, \qquad dZ_t^T = dZ_t, \tag{2.27}$$

are three independent Brownian motions under the T-forward measure Q^T. Therefore, the dynamics of r_t, v_t and S_t under Q^T are given by

$$dS_t = S_{t-} r_t dt + \sigma S_{t-} \left(\rho_v dZ_{T_t}^{vT} + \sqrt{1-\rho_v^2} dZ_{T_t}^T \right) + S_{t-} dJ_{T_t}^*, \tag{2.28}$$

$$dr_t = \left(\alpha(t) - \beta r_t - \frac{\sigma_r^2}{\beta} \left(1 - e^{-\beta(T-t)} \right) \right) dt + \sigma_r dZ_t^{rT}, \tag{2.29}$$

$$dv_t = \gamma(1 - v_t)dt + \sigma_v \sqrt{v_t} dZ_t^{vT}. \tag{2.30}$$

3. The pricing of a European call option on the given asset

Let $(S_t)_{t\in[0,T]}$ be the price of a financial asset modeled as a stochastic process on a filtered probability space $\left(\Omega, F, F_t, Q^T\right)$, and F_t is usually taken to be the price history up to time t. All processes in this section will be defined in this space. We denote C the price at time t of a European call option on the current price of an underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European option on the underlying stock S_t with strike price K is

$$\max(S_T - K, 0). \tag{3.1}$$

This means the holder will exercise his right only $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \le K$ then the holder will buy the underlying asset from the market and the value of the option is zero.

We would like to find a formula for pricing a European call option with strike price K and maturity T based on the model (2.28)–(2.30). Consider a continuous-time economy where interest rates are stochastic and the price of the European call option at time t under the T-forward measure Q^T is

$$\begin{aligned} C(t, S_t, r_t, v_t; T, K) &= P^*(t,T) E_{Q^T} \left(\max\left(S_T - K, 0\right) | S_t, r_t, v_t\right) \\ &= P^*(t,T) \int_0^\infty \max\left(S_T - K, 0\right) p_{Q^T}(S_T | S_t, r_t, v_t) dS_T. \end{aligned}$$

Here E_{Q^T} is the expectation with respect to the T-forward probability measure, p_{Q^T} is the corresponding conditional density given (S_t, r_t, v_t), and P^* is a zero coupon bond which is defined by

$$P^*(t,T) := E_{Q^T} \left[\exp\left(-\int_t^T r_s ds \right) | F_t \right]. \tag{3.2}$$

With a change in variable $X_t = \ln S_t$,

$$\begin{aligned} C(t, S_t, r_t, v_t; T, K) &= P^*(t,T) \int_{-\infty}^\infty \max\left(e^{X_T} - K, 0\right) p_{Q^T}\left(X_T | X_t, r_t, v_t, \right) dX_T \\ &= e^{X_t} P_1\left(t, X_t, r_t, v_t; T, K\right) - K P^*(t,T) P_2\left(t, X_t, r_t, v_t; T, K\right) \\ &= e^{X_t} \Pr\left(X_T > \ln K | X_t, r_t, v_t\right) - K P^*(t,T) \Pr\left(X_T > \ln K | X_t, r_t, v_t\right), \end{aligned} \tag{3.3}$$

where those probabilities in Eq. (3.3) are calculated under the probability measure Q^T.

The European call option for log asset price $X_t = \ln S_t$, will be denoted by

$$\hat{C}(t, X_t, r_t, v_t; T, \kappa) = e^{X_t}\tilde{P}_1(t, X_t, r_t, v_t; T, \kappa) - e^{\kappa}P^*(t, T)\tilde{P}_2(t, X_t, r_t, v_t; T, \kappa), \tag{3.4}$$

where $\kappa = \ln K$ and $\tilde{P}_j(t, X_t, r_t, v_t; T, \kappa) := P_j(t, X_t, r_t, v_t; T, K), j = 1, 2$.

Note that we do not have a closed form solution for these probabilities. However, these probabilities are related to characteristic functions which have closed form solutions as will be seen in Lemma 4.

Next, consider a continuous-time economy where interest rates are stochastic and satisfy Eq. (2.29). Since the SDE in Eq. (2.29) satisfies all the necessary conditions of Theorem 32, see Protter (2005, p. 238), then the solution has Markov property. As a consequence, the zero coupon bond price at time t under the forward measure Q^T in Eq. (3.2) satisfies

$$P^*(t, T) = E_{Q^T}\left[\exp\left(-\int_t^T r_s ds\right)|r_t\right]. \tag{3.5}$$

Note that $P^*(t, T)$ depends on r_t then it becomes a function $F(t, r_t)$ of r_t. This means that the calculation of $P^*(t, T)$ can now be formulated as a search for the function $F(t, r_t)$.

Lemma 2. *The price of a zero coupon bond can be derived by computing the expectation (3.5). We obtain*

$$P^*(t, T) = \exp(a(t, T) + b(t, T)r_t) \tag{3.6}$$

where $b(t, T) = \frac{1}{\beta}\left(e^{-\beta(T-t)} - 1\right)$, $a(t, T) = -f(0, t)b(t, T) + \ln\left(\frac{P^*(0,T)}{P^*(0,t)}\right) - \frac{3\sigma_r^2}{4\beta}\left[b(t, T)^2\left(1 - e^{-2\beta t}\right)\right]$.

Proof. Under the T-forward measure Q^T, the interest rate is given by Eq. (2.29). The specification of the interest rate means that the model (2.29) belongs to the affine class of interest rate models. Thus the bond price at time t with maturity T is of the form Eq. (3.6) where $a(t, T)$ and $b(t, T)$ are functions to be determined under the condition $a(T, T) = 0$ and $b(T, T) = 0$. We will now find explicit formulas for the functions $a(t, T)$ and $b(t, T)$ in Eq. (3.6).

The zero coupon bond price PDE satisfies (the proof is similar to Privault (2008, Proposition 4.1))

$$\frac{\partial F(t, r_t)}{\partial t} + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right)\frac{\partial F(t, r_t)}{\partial r_t} + \frac{1}{2}\frac{\partial^2 F(t, r_t)}{\partial r_t^2}\sigma_r^2 - r_t F(t, r_t) = 0. \tag{3.7}$$

Note that $F(t, r_t) = P^*(t, T)$. We substitute the value $F(t, r_t)$ from (3.6) into the above equation and after canceling some common factors, we have

$$\left(\frac{\partial a(t, T)}{\partial t} + r_t\frac{\partial b(t, T)}{\partial t}\right) + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) - \beta r_t\right)b(t, T) + \frac{1}{2}b^2(t, T)\sigma_r^2 - r_t = 0.$$

We can reduce it to two ordinary differential equations

$$\frac{\partial a(t, T)}{\partial t} + \frac{\sigma_r^2}{2}b^2(t, T) + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)b(t, T) = 0, \tag{3.8}$$

$$\frac{\partial b(t, T)}{\partial t} - \beta b(t, T) - 1 = 0, \tag{3.9}$$

with boundary conditions $a(T, T) = 0$, $b(T, T) = 0$.

Firstly, we note that the solution of Eq. (3.9) which satisfies the boundary conditions $b(T, T) = 0$ is

$$b(t, T) = \frac{1}{\beta}\left(e^{-\beta(T-t)} - 1\right). \tag{3.10}$$

Secondly, we try to solve Eq. (3.8). Note that

$$\int_t^T \frac{\partial a(u, T)}{\partial u}du = [a(u, T)]_{u=t}^{u=T} = a(T, T) - a(t, T) = -a(t, T). \tag{3.11}$$

Thus

$$a(t, T) = \left(\frac{3\sigma_r^2}{2}\right)\int_t^T (b(u, T))^2 du + \int_t^T \alpha(u)b(u, T)du. \tag{3.12}$$

It follows from Eqs. (2.9) and (3.6) that the forward rate at time 0 with the maturity T can be written as

$$f(0, T) = -\frac{\partial}{\partial T}\ln P^*(0, T) = -\frac{\partial a(0, T)}{\partial T} - r_0\frac{\partial b(0, T)}{\partial T}. \tag{3.13}$$

Differentiate $a(0, T)$ with respect to T and using $a(T, T) = 0$, $b(T, T) = 0$, we obtain from Eq. (3.12) that

$$\frac{\partial a(0,T)}{\partial T} = 3\sigma_r^2 \int_0^T b(u,T)\frac{\partial b(u,T)}{\partial T}du + \int_0^T \alpha(u)\frac{\partial b(u,T)}{\partial T}du.$$

Substituting the value of $b(u, T)$ from Eq. (3.10) into the above equation and after some calculations, we get

$$\frac{\partial a(0,T)}{\partial T} = \frac{\left(3\sigma_r^2\right)}{2\beta^2}\left(e^{-\beta T}-1\right)^2 - \int_0^T e^{-\beta(T-u)}\alpha(u)du.$$

Now substitute the value of $\frac{\partial a(0,T)}{\partial T}$ and the value of $\frac{\partial b(0,T)}{\partial T}$ into Eq. (3.13), we have

$$f(0,T) = -\frac{3\sigma_r^2}{2\beta^2}\left(e^{-\beta T}-1\right)^2 + \int_0^T e^{-\beta(T-u)}\alpha(u)du + r_0\left(e^{-\beta T}\right). \tag{3.14}$$

To isolate $\alpha(T)$, we differentiate $f(0, T)$ with respect to T and get

$$\frac{\partial f(0,T)}{\partial T} = \frac{3\sigma_r^2}{\beta}\left(e^{-2\beta T}-e^{-\beta T}\right) - r_0\beta e^{-\beta T} + \left(\alpha(T) - \beta\left(\int_0^T e^{-\beta(T-u)}\alpha(u)du\right)\right).$$

Using Eq. (3.14) to rewrite the above equation and after simplifying, we get

$$\alpha(T) = \frac{\partial f(0,T)}{\partial T} + \beta f(0,T) - \frac{3\sigma_r^2}{2\beta}\left(e^{-2\beta T}-1\right). \tag{3.15}$$

Next, we shall find a formula for $a(t, T)$ in Eq. (3.12). Note that

$$\frac{3\sigma_r^2}{2}\int_t^T b(u,T)^2du = \frac{3\sigma_r^2}{2\beta}\left(-\frac{1}{2}b(t,T)^2 + \frac{1}{\beta}\left(b(t,T)+T-t\right)\right),$$

and

$$\begin{aligned}\int_t^T \alpha(u)b(u,T)du &= \int_t^T \left(\frac{\partial f(0,T)}{\partial T} + \beta f(0,T) - \frac{3\sigma_r^2}{2\beta}\left(e^{-2\beta T}-1\right)\right)b(u,T)du \\ &= -f(0,t)b(t,T) - \int_t^T f(0,u)du - \frac{3\sigma_r^2}{2\beta^2}(T-t) \\ &\quad + \frac{3\sigma_r^2}{4\beta^3}\left[e^{-2\beta T} - 2e^{-\beta(T+t)} - 2e^{-\beta(T-t)} + e^{-2\beta t} + 2\right].\end{aligned}$$

Therefore

$$a(t,T) = -f(0,t)b(t,T) - \int_t^T f(0,u)du - \frac{3\sigma_r^2}{4\beta}b^2(t,T)(1-e^{-2\beta t}).$$

By definition, $P^*(0,T) = e^{-\int_0^T f(0,u)du}$. Thus $-\int_t^T f(0,u)du = \ln\left(\frac{P^*(0,T)}{P^*(0,t)}\right)$.

Finally, we have

$$a(t,T) = -f(0,t)b(t,T) + \ln\left(\frac{P^*(0,T)}{P^*(0,t)}\right) - \frac{3\sigma_r^2}{4\beta}b^2(t,T)(1-e^{-2\beta t}).$$

The proof is now complete. □

The following lemma shows the relationship between $\tilde{P}_1$ and $\tilde{P}_2$ in the option value of Eq. (3.4).

Lemma 3. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ in the option values of Eq. (3.4) satisfy the PIDEs:*

$$\begin{aligned} 0 \;=\; & \frac{\partial \tilde{P}_1}{\partial t} + A[P_1] + \frac{1}{2}\sigma^2 v\frac{\partial \tilde{P}_1}{\partial x} + \rho_v\sigma v\sigma_v\frac{\partial \tilde{P}_1}{\partial v} \\ & + v\int_{-\infty}^{\infty}\left[(e^y-1)\left(\tilde{P}_1(t,x+y,r,v;T,\kappa) - \tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]k(y)dy \end{aligned} \tag{3.16}$$

and subject to the boundary condition at expiration $t = T$,

$$\tilde{P}_1(T,x,r,v;T,\kappa) = 1_{x>\kappa}. \tag{3.17}$$

Moreover, $\tilde{P}_2$ satisfies the equation

$$\begin{aligned} 0 \;=\; & \frac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] - \frac{\sigma^2 v}{2}\frac{\partial \tilde{P}_2}{\partial x} + \sigma_r^2 b(t,T)\frac{\partial \tilde{P}_2}{\partial r} + \left(\frac{\partial a(t,T)}{\partial t} + r\frac{\partial b(t,T)}{\partial t}\right)\tilde{P}_2 \\ & + \left(\frac{3\sigma_r^2}{2} b^2(t,T) - r + (\alpha(t) - \beta r)\, b(t,T)\right)\tilde{P}_2 \end{aligned} \tag{3.18}$$

and subject to the boundary condition at expiration $t = T$,

$$\tilde{P}_2(T, x, r, v; T, \kappa) = 1_{x>\kappa}. \tag{3.19}$$

Here, for $i = 1, 2$,

$$\begin{aligned} A[\tilde{P}_i] = \; & r\frac{\partial \tilde{P}_i}{\partial x} + \left(\alpha(t) - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial \tilde{P}_i}{\partial r} + \gamma(1 - v)\frac{\partial \tilde{P}_i}{\partial v} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_i}{\partial v^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_i}{\partial x^2} \\ & + \frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_i}{\partial r^2} + (\rho_v \sigma v \sigma_v)\frac{\partial \tilde{P}_i}{\partial v \partial x} \\ & + v\int_{-\infty}^{\infty}\left(\tilde{P}_i(t, x+y, r, v; T, \kappa) - \tilde{P}_i(x, t, r, v; T, \kappa) - \left(\frac{\partial \tilde{P}_i}{\partial x}\right)(e^y - 1)\right) k(y)dy. \end{aligned} \tag{3.20}$$

Note that $1_{x>\kappa} = 1$ if $x > \kappa$ and zero otherwise. We assume that the jump kernel $k(y)$ exists.

Proof. See Appendix. □

4. The closed-form solution for European call options

For $j = 1, 2$, the characteristic functions for $\tilde{P}_j(t, x, r, v; T, \kappa)$, with respect to the variable κ, are defined by

$$f_j(t, x, r, v; T, u) := -\int_{-\infty}^{\infty} e^{iu\kappa}\, d\tilde{P}_j(t, x, r, v; T, \kappa), \tag{4.1}$$

with a minus sign to account for the negativity of the measure $d\tilde{P}_j$. Note that f_j also satisfies similar PIDEs

$$\frac{\partial f_j}{\partial t} + A_j\left[f_j\right](t, x, r, v; T, \kappa) = 0, \tag{4.2}$$

with the respective boundary conditions

$$f_j(T, x, r, v; T, u) = -\int_{-\infty}^{\infty} e^{iu\kappa}\, d\tilde{P}_j(t, x, r, v; T, \kappa) = -\int_{-\infty}^{\infty} e^{iu\kappa}(-\delta(\kappa - x))d\kappa = e^{iux}.$$

The following lemma shows how to calculate the characteristic functions for $\tilde{P}_1$ and $\tilde{P}_2$ as they appeared in Lemma 3.

Lemma 4. *The functions $\tilde{P}_1$ and $\tilde{P}_2$ can be calculated by the inverse Fourier transformations of the characteristic function, i.e.*

$$\tilde{P}_j(t, x, r, v; T, \kappa) = \frac{1}{2} + \frac{1}{\pi}\int_{0+}^{\infty} Re\left[\frac{e^{iu\kappa} f_j(t, x, r, v; T, u)}{iu}\right] du,$$

for $j = 1, 2$, with $Re[\cdot]$ denoting the real component of a complex number.

By letting $\tau = T - t$, the characteristic function f_j is given by

$$f_j(t, x, r, v; t+\tau, u) = \exp\left(iux + B_j(\tau) + rC_j(\tau) + vE_j(\tau) - (j-1)\ln P^*(t, t+\tau)\right),$$

where $\tilde{b}_{1j} = b_{2j} + \nabla_j$, $\tilde{b}_{2j} = b_{2j} - \nabla_j$, $\nabla_j = \sqrt{{b_{2j}}^2 - 4b_{0j}b_1}$, $b_{22} = \rho_v \sigma \sigma_v iu - \gamma$,

$$b_1 = \frac{\sigma_v^2}{2}, \qquad b_{21} = \rho_v \sigma \sigma_v (1 + iu) - \gamma, \qquad \alpha(t) = \frac{\partial f(0,t)}{\partial t} + \beta f(0,t) - \frac{3\sigma_r^2}{2\beta}\left(e^{-2\beta t} - 1\right),$$

$$b_{01} = \frac{\sigma^2}{2}\left(iu - u^2\right) + \int_{-\infty}^{\infty}\left[e^{iux+y} - iu(e^y - 1)\right]k(y)dy, \qquad C_1(\tau) = \frac{iu}{\beta}(1 - e^{-\beta\tau}),$$

$$C_2(\tau) = \frac{iu - 1}{\beta}\left(1 - e^{\beta\tau}\right),$$

$$b_{02} = -\frac{\sigma^2}{2}(iu+u^2) + \int_{-\infty}^{\infty}\left(e^{iux} - iu(e^y-1)\right)k(y)dy, \qquad E_j(\tau) = \frac{\left(e^{\tau\nabla_j}-1\right)\tilde{b}_{1j}\tilde{b}_{2j}}{2b_1\left(\tilde{b}_{1j} - e^{\tau\nabla_j}\tilde{b}_{2j}\right)},$$

$$\begin{aligned} B_1(\tau) &= \int_{T-\tau}^{T}\alpha(t)C_1(T-t)dt + \frac{\sigma_r^2}{2\beta^3}\left(\frac{u^2}{2}+\frac{iu}{\beta}\right)\left(\left(e^{\beta\tau}-2\right)^2 + 2\beta\tau - 1\right) \\ &\quad + \frac{\gamma^2\left(\tilde{b}_2 - \tilde{b}_1\right)}{2b_1}\ln\left(\frac{\tilde{b}_{21}-\tilde{b}_{11}}{e^{\tau\nabla}\tilde{b}_{21}-\tilde{b}_{11}}\right) + \frac{\gamma^2\tilde{b}_{21}\nabla_1\tau}{2b_1}, \\ B_2(\tau) &= \int_{T-\tau}^{T}\alpha(t)C_2(T-t)dt + \frac{\sigma_r^2\,(iu-1)}{2\beta^3}\left(\frac{1}{\beta} - \frac{(iu-1)}{2}\right)\left(\left(e^{\beta\tau}-2\right)^2 + 2\beta\tau - 1\right) + \frac{\gamma^2\tilde{b}_{22}\nabla_2\tau}{2b_1} \\ &\quad + \frac{\gamma^2\left(\tilde{b}_{22} - \tilde{b}_{12}\right)}{2b_1}\ln\left(\frac{\tilde{b}_{22}-\tilde{b}_{12}}{e^{\tau\nabla}\tilde{b}_{22}-\tilde{b}_{12}}\right). \end{aligned}$$

Proof. To solve the characteristic function explicitly, letting $\tau = T - t$ be the time-to-go, we conjecture that the function f_1 is given by

$$f_1(t, x, r, v; t+\tau, u) = \exp\left(iux + B_1(\tau) + rC_1(\tau) + vE_1(\tau)\right), \tag{4.3}$$

and the boundary condition $B_1(0) = C_1(0) = E_1(0) = 0$. This conjecture exploits the linearity of the coefficient in PIDEs (4.2). Note that the characteristic function of f_1 always exists. In order to substitute Eq. (4.3) into (4.2), firstly, we compute

$$\begin{aligned} &\frac{\partial f_1}{\partial t} = -\left(B_1'(\tau) + rC_1'(\tau) + vE_1'(\tau)\right)f_1, \qquad \frac{\partial f_1}{\partial x} = iuf_1, \qquad \frac{\partial f_1}{\partial r} = C_1(\tau)f_1, \qquad \frac{\partial f_1}{\partial v} = E_1(\tau)f_1, \\ &\frac{\partial^2 f_1}{\partial x^2} = -u^2 f_1, \qquad \frac{\partial^2 f_1}{\partial v^2} = E_1^2(\tau)f_1, \qquad \frac{\partial^2 f_1}{\partial r^2} = C_1^2(\tau)f_1, \qquad \frac{\partial^2 f_1}{\partial v\partial x} = iuE_1(\tau)f_1, \\ &f_1(t, x+y, r, v; t+\tau, u) - f_1(t, x, r, v; t+\tau, u) = e^{iux}f_1(t, x, r, v; t+\tau, u). \end{aligned}$$

Substituting all the above terms into Eq. (4.2), after canceling the common factor of f_1, we get a simplified form as follows:

$$\begin{aligned} 0 &= v\left(-E_1'(\tau) + \frac{\sigma_v^2}{2}E_1^2(\tau) + \left(\rho_v\sigma\sigma_v\,(1+iu) - \gamma\right)E_1(\tau) + \frac{\sigma^2\left(iu-u^2\right)}{2}\right. \\ &\quad \left. + \int_{-\infty}^{\infty}\left[e^{iux+y} - iu(e^y-1)\right]k(y)dy\right) + \left(-B_1'(\tau) + \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)C_1(\tau)\right. \\ &\quad \left. +\ \gamma E_1(\tau) + \frac{\sigma_r^2}{2}C_1^2(\tau)\right) + r\left(-C_1'(\tau) + iu - \beta C_1(\tau)\right). \end{aligned}$$

By separating the order r, v and ordering the remaining terms, we can reduce it to three ordinary differential equations (ODEs) as follows:

$$C_1'(\tau) = -\beta C_1(\tau) + iu, \tag{4.4}$$

$$E_1'(\tau) = \frac{\sigma_v^2}{2}E_1^2(\tau) + \left[\rho_v\sigma\sigma_v\,(1+iu) - \gamma\right]E_1(\tau) + \frac{\sigma^2(iu-u^2)}{2} + \int_{-\infty}^{\infty}\left(e^{iux+y} - iu(e^y-1)\right)k(y)dy, \tag{4.5}$$

$$B_1'(\tau) = \left(\alpha(t) - \frac{\sigma_r^2}{\beta}\left(1-e^{-\beta(T-t)}\right)\right)C_1(\tau) + \frac{\sigma_r^2}{2}C_1^2(\tau) + \gamma E_1(\tau). \tag{4.6}$$

It is clear from Eq. (4.4) and $C(0) = 0$ that

$$C_1(\tau) = \frac{iu}{\beta}(1 - e^{-\beta\tau}). \tag{4.7}$$

Let $b_0 = \frac{\sigma^2(iu-u^2)}{2} + \int_{-\infty}^{\infty}\left[e^{iux+y} - iu(e^y-1)\right]k(y)dy$, $b_1 = \frac{\sigma_v^2}{2}$ and $b_2 = \left(\rho_v\sigma\sigma_v\,(1+iu) - \gamma\right)$.

Substitute these constants into Eq. (4.5), one gets

$$\begin{aligned} E_1'(\tau) &= b_1\left(E_1^2(\tau) + \frac{b_2}{b_1}E_1(\tau) + \frac{b_0}{b_1}\right) \\ &= b_1\left[\left(E_1(\tau) - \frac{-b_2 + \sqrt{b_2{}^2 - 4b_0b_1}}{2b_1}\right)\left(E_1(\tau) - \frac{-b_2 - \sqrt{b_2{}^2 - 4b_0b_1}}{2b_1}\right)\right]. \end{aligned}$$

By the method of variable separation, we have

$$\frac{dE_1(\tau)}{\left(E_1(\tau)-\frac{-b_2+\sqrt{{b_2}^2-4b_0b_1}}{2b_1}\right)\left(E_1(\tau)-\frac{-b_2-\sqrt{{b_2}^2-4b_0b_1}}{2b_1}\right)}=b_1d\tau.$$

Using partial fraction on the left hand side, one obtains

$$\left(\frac{1}{\left(E_1(\tau)-\frac{-b_2+\sqrt{{b_2}^2-4b_0b_1}}{2b_1}\right)}-\frac{1}{\left(E_1(\tau)-\frac{-b_2-\sqrt{{b_2}^2-4b_0b_1}}{2b_1}\right)}\right)dE_1(\tau)=\sqrt{{b_2}^2-4b_0b_1}d\tau.$$

Integrating both sides, we obtain

$$\ln\left(\frac{E_1(\tau)-\frac{-b_2+\sqrt{{b_2}^2-4b_0b_1}}{2b_1}}{E_1(\tau)-\frac{-b_2-\sqrt{{b_2}^2-4b_0b_1}}{2b_1}}\right)=\tau\sqrt{{b_2}^2-4b_0b_1}+E_0.$$

Applying boundary condition $E_1(\tau=0)=0$, we get $E_0=\ln\left(\frac{-b_2+\sqrt{{b_2}^2-4b_0b_1}}{-b_2-\sqrt{{b_2}^2-4b_0b_1}}\right)$.

Solving for E_1, we have

$$E_1(\tau)=\frac{\left(e^{\tau\sqrt{{b_2}^2-4b_0b_1}}-1\right)\tilde{b}_1\tilde{b}_2}{2b_1\left(\tilde{b}_1-e^{\tau\sqrt{{b_2}^2-4b_0b_1}}\tilde{b}_2\right)},$$

where $\tilde{b}_1=b_2+\sqrt{{b_2}^2-4b_0b_1}$, and $\tilde{b}_2=b_2-\sqrt{{b_2}^2-4b_0b_1}$.

In order to solve $B_1(\tau)$, we substitute $C_1(\tau)$ and $E_1(\tau)$ into Eq. (4.6) to get

$$\begin{aligned}B_1(\tau)&=\int_{T-\tau}^{T}\alpha(t)C_1(T-t)dt+\frac{{\sigma^2}_r}{2\beta^3}\left(\frac{u^2}{2}+\frac{iu}{\beta}\right)\left(\left(e^{\beta\tau}-2\right)^2+2\beta\tau-1\right)\\&\quad+\frac{\gamma^2\left(\tilde{b}_2-\tilde{b}_1\right)}{2b_1}\ln\left(\frac{\tilde{b}_2-\tilde{b}_1}{e^{\tau\nabla}\tilde{b}_2-\tilde{b}_1}\right)+\frac{\gamma^2\tilde{b}_2\nabla\tau}{2b_1},\end{aligned}$$

where $\nabla=\sqrt{{b_2}^2-4b_0b_1}$ and $\alpha(t)$ is defined in Eq. (3.15).

The details of the proof for the characteristic function f_2 are similar to f_1. Hence, we have

$$f_2(t,x,r,v;t+\tau,u)=\exp\left[iux+B_2(\tau)+rC_2(\tau)+vE_2(\tau)-\ln P^*(t,t+\tau)\right],$$

where $B_2(\tau)$, $C_2(\tau)$, and $E_2(\tau)$ are as given in the lemma.

Up to this point, we obtained the characteristic functions in close form. However, we are interested in the probability $\tilde{P}_j$. These can be inverted from the characteristic functions by performing the following integration

$$\tilde{P}_j(t,x,r,v;T,\kappa)=\frac{1}{2}+\frac{1}{\pi}\int_{0+}^{\infty}Re\left(\frac{e^{iu\kappa}f_j(t,x,v\cdot r;T,u)}{iu}\right)du,\quad j=1,2,\tag{4.8}$$

where $X_t=\ln S_t$ and $\kappa=\ln K$ (see Sattayatham & Intarasit, 2011).

The proof is now complete. □

In summary, we have just proved the following main theorem.

Theorem 5. *The value of a European call option of SDE (2.28) is*

$$C(t,S_t,r_t,v_t;T,K)=S_t\tilde{P}_1\left(t,X_t,r_t,v_t;T,\kappa\right)-KP^*(t,T)\tilde{P}_2\left(t,X_t,r_t,v_t;T,\kappa\right)$$

where $\tilde{P}_1$ and $\tilde{P}_2$ are given in Lemma 4 and $P^*(t,T)$ is given in Lemma 2.

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Appendix. Proof of Lemma 3

By Ito's lemma, $\hat{C}(t,x,r,v)$ follows the partial integro–differential equation (PIDE)

$$\begin{aligned} r\hat{C} = {} & \frac{\partial \hat{C}}{\partial t} + \left(r - \frac{1}{2}\sigma^2 v\right)\frac{\partial \hat{C}}{\partial x} + \left(\alpha(t) - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial \hat{C}}{\partial r} + \gamma(1-v)\frac{\partial C}{\partial v} \\ & + \frac{\sigma_r^2}{2}\frac{\partial^2 \hat{C}}{\partial r^2} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \hat{C}}{\partial v^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \hat{C}}{\partial x^2} + (\rho_v \sigma v \sigma_v)\frac{\partial^2 \hat{C}}{\partial x \partial v} \\ & + v\int_{-\infty}^{\infty}\left(\hat{C}(t,x+y,r,v) - \hat{C}(t,x,r,v) - \frac{\partial \hat{C}}{\partial x}(e^y - 1)\right)k(y)dy. \end{aligned} \tag{A.1}$$

We plan to substitute Eq. (3.4) into (A.1). Firstly, we compute

$$\frac{\partial \hat{C}}{\partial t} = e^x\frac{\partial \tilde{P}_1}{\partial t} - e^\kappa P^*(t,T)\left[\frac{\partial \tilde{P}_2}{\partial t} + \tilde{P}_2\frac{\partial}{\partial t}\left(a(t,T) + b(t,T)r\right)\right],$$

$$\frac{\partial \hat{C}}{\partial x} = e^x\left(\frac{\partial \tilde{P}_1}{\partial x} + \tilde{P}_1\right) - e^\kappa P^*(t,T)\frac{\partial \tilde{P}_2}{\partial x},$$

$$\frac{\partial \hat{C}}{\partial v} = e^x\frac{\partial \tilde{P}_1}{\partial v} - e^\kappa P^*(t,T)\frac{\partial \tilde{P}_2}{\partial v}, \qquad \frac{\partial \hat{C}}{\partial r} = e^x\frac{\partial \tilde{P}_1}{\partial r} - e^\kappa P^*(t,T)\left(\frac{\partial \tilde{P}_2}{\partial r} + \tilde{P}_2 b(t,T)\right),$$

$$\frac{\partial^2 \hat{C}}{\partial x^2} = e^x\left(\frac{\partial^2 \tilde{P}_1}{\partial x^2} + 2\frac{\partial \tilde{P}_1}{\partial x} + \tilde{P}_1\right) - e^\kappa P^*(t,T)\frac{\partial^2 \tilde{P}_2}{\partial x^2}, \qquad \frac{\partial^2 \hat{C}}{\partial v^2} = e^x\frac{\partial^2 \tilde{P}_1}{\partial v^2} - e^\kappa P^*(t,T)\frac{\partial^2 \tilde{P}_2}{\partial v^2},$$

$$\frac{\partial^2 \hat{C}}{\partial r^2} = e^x\frac{\partial^2 \tilde{P}_1}{\partial r^2} - e^\kappa P^*(t,T)\left(\frac{\partial^2 \tilde{P}_2}{\partial r^2} + 2b(t,T)\frac{\partial \tilde{P}_2}{\partial r} + \tilde{P}_2 b^2(t,T)\right),$$

$$\frac{\partial^2 \hat{C}}{\partial v \partial x} = e^x\left(\frac{\partial \tilde{P}_1}{\partial v \partial x} + \frac{\partial \tilde{P}_1}{\partial v}\right) - e^\kappa P^*(t,T)\frac{\partial \tilde{P}_2}{\partial v \partial x},$$

$$\begin{aligned} & \hat{C}(t,x+y,r,v;T,\kappa) - \hat{C}(t,x,r,v;T,\kappa) \\ & \quad = e^x\left[(e^y - 1)\tilde{P}_1(t,x+y,r,v;T,\kappa) + \left(\tilde{P}_1(t,x+y,r,v;T,\kappa) - \tilde{P}_1(x,t,r,v;T,\kappa)\right)\right] \\ & \qquad - e^\kappa P^*(t,T)\left[\tilde{P}_2(t,x+y,r,v;T,\kappa) - \tilde{P}_2(t,x,r,v;T,\kappa)\right]. \end{aligned}$$

Substitute all terms above into Eq. (A.1) and separate it by assumed independent terms of $\tilde{P}_1$ and $\tilde{P}_2$. This gives two PIDEs for the forward probability for $\tilde{P}_i(t,x,r,v;T,\kappa)$, $j = 1, 2$:

$$\begin{aligned} 0 = {} & \frac{\partial \tilde{P}_1}{\partial t} + \left(r + \frac{1}{2}\sigma^2 v\right)\frac{\partial \tilde{P}_1}{\partial x} + \left[\gamma(1-v) + (\rho_v \sigma v \sigma_v)\right]\frac{\partial \tilde{P}_1}{\partial v} + \frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_1}{\partial r^2} \\ & + \left(\alpha(t) - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right)\right)\frac{\partial \tilde{P}_1}{\partial r} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_1}{\partial v^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_1}{\partial x^2} + (\rho_v \sigma v \sigma_v)\frac{\partial \tilde{P}_1}{\partial v \partial x} \\ & + v\int_{-\infty}^{\infty}\left[\tilde{P}_1(t,x+y,r,v;T,\kappa) - \tilde{P}_1(x,t,r,v;T,\kappa) - \left(\frac{\partial \tilde{P}_1}{\partial x}\right)(e^y - 1)\right]k(y)dy \\ & + v\int_{-\infty}^{\infty}\left[(e^y - 1)\left(\tilde{P}_1(t,x+y,r,v;T,\kappa) - \tilde{P}_1(x,t,r,v;T,\kappa)\right)\right]k(y)dy, \end{aligned} \tag{A.2}$$

and subject to the boundary condition at the expiration time $t = T$ according to Eq. (3.17).

By using the notation in Eq. (3.20), Eq. (A.2) becomes

$$\begin{aligned} 0 = {} & \frac{\partial \tilde{P}_1}{\partial t} + A[P_1] + \left(\frac{1}{2}\sigma^2 v\right)\tilde{P}_1 + \sigma^2 v\frac{\partial \tilde{P}_1}{\partial x} + \rho_v \sigma v \sigma_v\frac{\partial \tilde{P}_1}{\partial v} \\ & + v\int_{-\infty}^{\infty}\left[\tilde{P}_1(t,x+y,r,v;T,\kappa) - \tilde{P}_1(x,t,r,v;T,\kappa) - \left(\frac{\partial \tilde{P}_1}{\partial x}\right)(e^y - 1)\right]k(y)dy := \frac{\partial \tilde{P}_1}{\partial t} + A_1[\tilde{P}_1]. \end{aligned}$$

For $\tilde{P}_2(t, x, r, v; T, \kappa)$:

$$
\begin{aligned}
0 = {} & \frac{\partial \tilde{P}_2}{\partial t} + \left(r - \frac{1}{2}\sigma^2 v\right)\frac{\partial \tilde{P}_2}{\partial x} + \left(\alpha(t) - \beta r - \frac{\sigma_r^2}{\beta}\left(1 - e^{-\beta(T-t)}\right) + \sigma_r^2 b(t, T)\right)\frac{\partial \tilde{P}_2}{\partial r} \\
& + \gamma(1 - v)\frac{\partial \tilde{P}_2}{\partial v} + \frac{\sigma_v^2 v}{2}\frac{\partial^2 \tilde{P}_2}{\partial v^2} + \frac{\sigma_r^2}{2}\frac{\partial^2 \tilde{P}_2}{\partial r^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 \tilde{P}_2}{\partial x^2} \\
& + (\rho_v \sigma v \sigma_v)\frac{\partial \tilde{P}_2}{\partial v \partial x} + \left(\frac{3\sigma_r^2}{2} b^2(t, T) - r + (\alpha(t) - \beta r)\, b(t, T)\right)\tilde{P}_2 + \left(\frac{\partial a(t, T)}{\partial t} + r\frac{\partial b(t, T)}{\partial t}\right)\tilde{P}_2 \\
& + v\int_{-\infty}^{\infty}\left(\tilde{P}_2(t, x + y, r, v; T, \kappa) - \tilde{P}_2(t, x, r, v; T, \kappa) - \frac{\partial \tilde{P}_2}{\partial x}(e^y - 1)\right)k(y)dy, \qquad \text{(A.3)}
\end{aligned}
$$

and subject to the boundary condition at expiration time $t = T$ according to Eq. (3.19). Again, by using the notation (3.20), Eq. (A.3) becomes

$$
\begin{aligned}
0 = {} & \frac{\partial \tilde{P}_2}{\partial t} + A[\tilde{P}_2] - \frac{\sigma^2 v}{2}\frac{\partial \tilde{P}_2}{\partial x} + \sigma_r^2 b(t, T)\frac{\partial \tilde{P}_2}{\partial r} + \left(\frac{\partial a(t, T)}{\partial t} + r\frac{\partial b(t, T)}{\partial t}\right)\tilde{P}_2 \\
& + \left(\frac{3\sigma_r^2}{2} b^2(t, T) - r + (\alpha(t) - \beta r)\, b(t, T)\right)\tilde{P}_2 := \frac{\partial \tilde{P}_2}{\partial t} + A_2[\tilde{P}_2].
\end{aligned}
$$

The proof is now completed.

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