# รูปแบบค่าสินไหมทดแทนและการกำหนดเบี้ยประกันภัย
# โดยใช้การแจกแจงผสมอนันต์

**นางสาวศศิธร อนันตโสภณ**

**วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต**
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**มหาวิทยาลัยเทคโนโลยีสุรนารี**
**ปีการศึกษา 2558**

# AN INSURANCE CLAIM AND PRICING MODEL USING INFINITE MIXTURE DISTRIBUTIONS

**Sasithorn Anantasopon**

**A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Applied Mathematics**

**Suranaree University of Technology**

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# AN INSURANCE CLAIM AND PRICING MODEL USING INFINITE MIXTURE DISTRIBUTIONS

Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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ศศิธร อนันตโสภณ : รูปแบบค่าสินไหมทดแทนและการกำหนดเบี้ยประกันภัยโดยใช้การแจกแจงผสมอนันต์ (AN INSURANCE CLAIM AND PRICING MODEL USING INFINITE MIXTURE DISTRIBUTIONS) อาจารย์ที่ปรึกษา : ศาสตราจารย์ ดร.ไพโรจน์ สัตยธรรม, 116 หน้า.

งานวิจัยนี้มีวัตถุประสงค์เพื่อศึกษาการสร้างรูปแบบการแจกแจงผสมอนันต์ ของความสูญเสียด้านประกันวินาศภัยสำหรับข้อมูลรายเดี่ยวและนำรูปแบบที่ได้นั้นไปกำหนดเบี้ยประกันภัย ในการศึกษาครั้งนี้รูปแบบของความสูญเสียด้านประกันภัย ประกอบไปด้วย 2 ส่วนคือ ส่วนของการจำลอง และส่วนของการนำไปใช้ ซึ่งจะมีการอธิบายดังต่อไปนี้

ส่วนการจำลอง : กลุ่มตัวอย่างการทดลองจะถูกจำลองให้เป็นการแจกแจงของความสูญเสียแบบผสม ซึ่งประกอบไปด้วยการแจกแจงลอกนอร์มอล แกมม่า และไวบูลล์ ข้อมูลที่ใช้จำลองโดย MATLAB ซึ่งกระทำซ้ำกัน 250 ครั้ง การประมาณค่าพารามิเตอร์สำหรับรูปแบบของการแจกแจงแบบดั้งเดิม (classical distribution) และการแจกแจงแบบผสมอนันต์ (infinite mixture distribution) ใช้วิธีภาวะน่าจะเป็นสูงสุด (Maximum Likelihood Estimate : MLE) มีขนาดของกลุ่มตัวอย่างที่ผ่านการทดสอบว่าเป็นกลุ่มตัวอย่างที่เหมาะสม จำนวน 99 ตัวอย่าง สถิติที่ใช้ในการทดสอบกับกลุ่มตัวอย่างเหล่านี้คือ โคโมโกรอฟ-สไมร์นอฟ (Kolmogorov-Smirnov test : K-S test) ผลสรุปว่า ค่า $D-$ value ของการแจกแจงแบบผสมอนันต์ มีค่าความคลาดเคลื่อนน้อยกว่าเมื่อเปรียบเทียบกับการแจกแจงแบบดั้งเดิม

ส่วนของการนำไปใช้ : งานวิจัยนี้ใช้ข้อมูลรายเดี่ยวของการจ่ายค่าสินไหมทดแทนของการประกันภัยรถยนต์ในปี 2552 ของบริษัทประกันวินาศภัยแห่งหนึ่งในประเทศไทย จำนวน 1,296 ข้อมูล มีลักษณะสอดคล้องกับการแจกแจงแบบผสมอนันต์ ด้วยระดับความเชื่อมั่น 99%

การกำหนดราคาประกันภัย : การกำหนดราคาเบี้ยประกันภัย เราจะใช้ตัวแบบเชิงเส้นวางนัยทั่วไป (Generalized Linear Model : GLM) เมื่อตัวแปรตามอยู่ในรูปแบบการแจกแจงแบบผสมอนันต์ มี 3 รูปแบบ ประกอบไปโดย อายุ เพศ และอายุและเพศ โดยใช้ผลรวมค่าคลาดเคลื่อนสัมบูรณ์ (Sum of Absolute Error : SAE), ค่ากลางของความคลาดเคลื่อนสัมบูรณ์ (Mean Absolute Error : MAE) และค่ากลางของความคลาดเคลื่อนกำลังสอง (Mean Square Error : MSE) ผลการศึกษาพบว่า รูปแบบที่ประกอบไปด้วยอายุและเพศ ให้ค่าความคลาดเคลื่อนน้อยกว่า เมื่อเปรียบเทียบกับเฉพาะรูปแบบอายุ และเฉพาะรูปแบบเพศ ดังนั้นจะใช้ผลลัพธ์จากรูปแบบที่ประกอบไปด้วยอายุและเพศเพื่อที่จะคำนวณค่าเบี้ยประกันภัย โดยใช้การคูณของค่าเฉลี่ยสินไหม

ทดแทน (Claim Severity) และค่าเฉลี่ยความถี่ของค่าสินไหมทดแทน (Claim Frequency) ผลลัพธ์ค่าเบี้ยประกันภัยที่ได้ ก็จะให้ค่าเหมาะสมยุติธรรมของแต่ละบุคคล โดยไม่มีอิทธิพลจากตัวแปรอื่น

สาขาวิชาคณิตศาสตร์ ลายมือชื่อนักศึกษา ศศิธร อพันต์โสภณ

ปีการศึกษา 2558 ลายมือชื่ออาจารย์ที่ปรึกษา

SASITHORN ANANTASOPON : AN INSURANCE CLAIM AND PRICING MODEL USING INFINITE MIXTURE DISTRIBUTIONS. THESIS ADVISOR : PROF. PAIROTE SATTAYATHAM, Ph.D. 116 PP.

CLAIM SEVERITY/CLASSICAL DISTRIBUTIONS/INFINITE MIXTURE DISTRIBUTION/GENERALIZED LINEAR MODEL/PURE PREMIUM

The objective of this study is to construct a novel insurance claim model employing infinite mixture distributions for individual data, and use the model for pricing of insurance premiums. In this study, the insurance claim modeling consists of two parts, namely, Simulations and Application which are explained as follows :

Simulations : the sample groups are simulated by a combination of claim distributions which are Lognormal, Gamma and Weibull. Data sets were created using MATLAB with 250 iterations. The parameter estimation used for both, classical and infinite mixture distributions, is the Maximum Likelihood Estimate (MLE). Having tested sample size by running numerous combinations of claim distributions and data sizes, we found 99 combinations yielding optimum sample sizes. Hence, we introduced Kolmogorov-Smirnov test (K-S test) to match these samples with the classical and infinite mixture distributions. The $D-$values of the infinite mixture distributions showed lower errors, when compared with the classical distributions.

Application : Individual data of motor insurance claims for the year 2009 from a non-life insurance company in Thailand were matched to the infinite mixture distributions. The 1,296 observations could be fitted to an infinite mixture distribution at a confidence level is 99%.

Insurance Pricing : to price the insurance premium, the Generalized Linear Model (GLM) with response variables of infinite mixture distribution were utilized. Three models were employed, inducing age, gender, and age and gender, respectively. Evaluating Sum of Absolute Errors (SAE), Mean Absolute Errors (MAE) and Mean Square Errors (MSE), we found that the model incorporating both age and gender carries less error compared to the age model and the gender model individually. Then, we use the results from the age and gender model to calculate insurance premiums using multiplication of the means of Claim Severity and Claim Frequency. Finally, the premium outcome is a fair individual insurance premium, without interference.

| | |
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Sasithorn Anantasopon

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# ABBREVIATIONS AND SYMBOLS

| | |
|---|---|
| CDF | Cumulative Distribution Function |
| PDF | Probability Density Function |
| MLE | Maximum Likelihood Estimate |
| MGF | Moment Generating Function |
| i.i.d. | independent identically distributed |
| $X$ | random variable of claim (Claim Severity) |
| $Y$ | random variable of claim (Claim Frequency) |
| $n$ | number of policies (1,296) |
| $E[X]$ | Expected value of $X$ |
| $\mathrm{Var}[X]$ | Variance of $X$ |
| $LN(\mu,\sigma)$ | Lognormal distribution with parameters $\mu$ and $\sigma$ |
| $Exp(\theta)$ | Exponential distribution with parameter $\theta$ |
| $IExp(\theta)$ | Inverse Exponential distribution with parameter $\theta$ |
| $IPa(\tau,\theta)$ | Inverse Pareto distribution with parameters $\tau$ and $\theta$ |
| GLM | Generalized Linear Model |
| SAE | Sum of Absolute Errors |
| MAE | Mean Absolute Error |
| MSE | Mean Square Error |

# CHAPTER I

# INTRODUCTION

## 1.1 Introduction and Motivation

A primary attribute of the actuary has been the ability to successfully apply mathematical and statistical techniques to insurance claim data, both in its analysis and interpretation. The modeling of claims is an important task for claim estimation, since a good estimation of a claim leads to good Insurance Pricing. Therefore, we focus on two concerns of the actuary, which are Claim Modeling and Insurance Pricing.

Claim Modeling means the ability to predict claims as accurately as possible in order to estimate future company liabilities. There are two major methods to model claims which are the modeling of Claim Severity and of Claim Frequency. Claim Severity refers to the monetary claim on an insurance claim and is usually modeled as a non-negative continuous random variable using mixed distributions, (Tes, 2009). Whereas the Claim Frequency is the number of claims. The classical distributions can not be fitted to arbitrary claim data.

In general, mixture models incorporate finite or infinite mixture distributions. The finite mixture distribution is one of the methods used to obtain new probability distributions. In the statistical literature, the finite mixture models emerged in the 1960s and 1970s. They were used for modeling of unobserved heterogeneity in the population.

Many authors presented the modeling of finite mixture models, i.e., Mohamed, Ahmad and Noriszura (2010) who proposed a model of aggregate claims based on a compound Poisson-Pareto distribution. Moreover, a paper of Sattayatham and Talangtam (2012) presented finite mixture Lognormal distributions and applied the models to motor insurance claims data. Mauro *et al*. (2012) proposed finite mixture Skew Normal distributions and applied them to the insurance claim data set of Danish fire losses. Recently, Erisoglu, Servi, Erisoglu and Calis (2013) used two mixture gamma distributions for the estimation of heterogeneous wind data sets.

A finite mixture distribution is limited by the number of components (k), which depends on the mean clustering. In order to solve this problem, we are interested in employing infinite mixture distributions. One of the reasons for using an infinite mixture model is to obtain new probability distributions and work with unknown parameters which will be simpler than to work on finite mixture distributions. Infinite mixture distributions are described in Hogg, Craig and McKean (2005), Klugman *et al*. (2008) and Catherine, Merran, Nicholas and Brian (2011).

The modeling of claims leads to the pricing of insurance premiums. The Insurance Pricing concepts compose a priori and a posteriori pricing, (David, 2015). There are many other statistical methods, namely; expected value, standard deviation, variance, semi- variance, Wang Transform, Esscher Transform and etc.

Traditionally, the expected (average) claim is the most widely used measure to obtain the premium that is transferred from the insured or policyholder to the insurer. The average claim measure leads to assigning a single insurance premium rate. Such single rate insurance premiums are unfair for most customers since the risk factors of policyholders, i.e., driver's age, gender, marital status, type of car driven or vehicle's age, are different. To solve this problem, the insurance company should define different rates of insurance premiums with fairer premiums. Since the group samples for an insurance premium pricing have never agreed with the normal distribution, in this Thesis, we then employ the Generalized Linear Model (GLM) to analyse the sample cases. The GLM in general, has been developed from regression models using response and covariate variables. The response variables come from a distribution in the exponential family. Therefore, estimation of response variables according to the principle of GLM uses a link function which depends on the distribution of the response variables. Additionally, we introduce the Maximum Likelihood Estimate (MLE) to estimate the model parameters. To trace back, the early development of the GLM occurred in the 1970s and 1980s, and the GLM is well explained by McCullagh and Nelder (1989), Dobson (2002), Jonge and Heller (2008). Ohlsson and Johansson (2010) stipulated many important illustrations of how to use GLMs in non-life Insurance Pricing. Haberman and Renshaw (1996) reviewed the applications of generalized linear models to actuarial problems.

## 1.2 Historical Review

A substational number of Claim Models were derived by many authors who have investigated and discussed Claim Severity and constructed some new distributions using infinite mixture distributions. Frangos and Karlis (2004) investigated a model of Claim Size distribution which has Exponential-inverse Gaussian distribution. The model is fitted to car accident claims data which comes from a large Greek insurance company. Emilio *et al.* (2008) proposed a negative binomial inverse Gaussian distribution (NBIG) which is applied to automobile insurance. The NBIG distribution is preferred to the negative binomial and Poisson distribution for computing automobile insurance premiums. Recently, Pacakova and Zapletal (2013) proposed the Pareto distribution which is derived from the Exponential and Gamma distributions. This model provides a better fit to the claim amounts in compulsory third party liability of motor vehicles insured by some Czech insurance company.

In Insurance Pricing, many authors investigated the risk factors of automobile insurance for appropriate pricing, for example, Arthur (1994) used the GLM as a comprehensive modeling tool for the study of the claims process (Claim Frequency and Claim Severity) in the presence of covariates. In that context, he developed an application of the motor insurance claims experience for a recent calendar year, and later adopted by many leading U.K. insurance companies. Kart *et al*. (2000) explained how a dynamic pricing system can be built for personal line insurance by using the statistical technique of GLM for estimating the risk premiums. Roosvelt and Mostry (2004) proposed that the GLM model should be used for determining claim settlements and breaking down claim costs, according to the risk factors which provide a logical analysis. Geoff and Serhat (2007) discussed the most frequent mistakes made by companies beginning to build GLMs. Recently, Silvie and Lenka (2014) proposed an estimate of annual Claim Frequency for vehicle insurance based on GLM. The case study was based on 57,410 vehicles, and results confirm the importance of three factors, which are age group of the policyholder, vehicle age, and area of residence.

## 1.3 Objective and Overview of the Thesis

The objective of this study is to construct a novel insurance claim model employing infinite mixture distributions for individual data, and use the model for pricing of insurance premiums. In this study, the insurance claim modeling consists of two parts, namely, Simulations and Application. In this study, we employ GLM for stimulating infinite mixture distrition and use most likely estimstiom (MLE) to unveil parameter estimations.

The Thesis consists of five chapters. Chapter II presents the preliminaries and some of the mathematical and statistical background used in this Thesis. Chapter III proposes the Claim Model, which is constructed from an infinite mixture distribution. The MLE is provided for the estimation of the parameter of the distribution. We executed numerical experiments of sample groups to be fitted to the infinite mixture distribution. An application to observed data is given in this section. Chapter IV presents the construction of a GLM, at which the response variables are modeled by an infinite mixture distribution. A comparison of the results of the predicted values of Claim Severity from all possible risk factors is also represented in this section. The GLM has been applied to calculate the premium for the observed data. The conclusions, discussion, and further research are shown in the last chapter.

The next chapter explains the basic knowledge of experimental statistics which will be the fundament of the construction of the models in Chapter III and Chapter IV.

# CHAPTER II

# PRELIMINARIES

In this chapter, we introduce the definitions and theories of some of the mathematical and statistical material that will be useful for claim modeling and insurance pricing in this research study.

## 2.1 Events and Probability Theory

We review the definitions of events and probability theory which can be found in Brezeniak and Zastawniak (1999).

**Definition 2.1** Let $\Omega$ be a non-empty set. A $\sigma-$ *field* $\mathcal{F}$ on $\Omega$ is a family of subsets of $\Omega$ such that

1. the empty set $\phi \in \mathcal{F}$ ;
2. if $B$ belongs to $\mathcal{F}$, then so does the complement $\Omega/B$;
3. if $B_1, B_2, \ldots$ is a sequence of set in $\mathcal{F}$, then their union $B_1 \cup B_2 \cup \ldots$ also belongs to $\mathcal{F}$.

**Definition 2.2** Let $\mathcal{F}$ be a $\sigma-$ *field* on $\Omega$. A *probability measure* $P$ is a

$$P: \mathcal{F} \to [0,1]$$

such that

1. $P(\Omega) = 1$;

2. if $B_1, B_2, \ldots$ are pairwise disjoint sets (that is, $B_i \cap B_j = \phi$ for $i \neq j$ ) belonging to Ƒ, then $P(B_1 \cup B_2 \cup \ldots) = P(B_1) + P(B_2) + \ldots$

The triple (Ω,Ƒ,P) is called a *probability space*. A set belonging to Ƒ is called an events.

## 2.2 Random Variables

**Definition 2.3** If Ƒ is a $\sigma-$ *field* on Ω, then a function $X:\Omega \to \mathbb{R}$ is said to be Ƒ-measurable if

$$(X \in B) \in \mathcal{F}$$

for every Borel set $B \in \beta(\mathbb{R})$. If (Ω,Ƒ,P) is a probability space, then such a function $X$ is called *a random variable.*

**Definition 2.4** The $\sigma-$ *field* $\sigma(X)$ *generated by a random variable* $X:\Omega \to R$ consists of all sets of the form $(X \in B)$, where $B$ is a Borel set in $\mathbb{R}$.

**Definition 2.5** Every random variable $X:\Omega \to \mathbb{R}$ gives rise to a probability measure

$$P_X(B) = P(X \in B)$$

on $\mathbb{R}$ defined on the $\sigma-$ *field* of Borel sets $B \in \beta(\mathbb{R})$. We call $P_X$ the *distribution* of $X$.

## 2.3 Distribution Functions

We review the distribution function which can be found in Knight (1999). **Definition 2.6** Let $X$ be a random variable on the probability space (Ω,Ƒ,P). We define $F_X:\mathbb{R} \to [0,1]$ by

$$F_X(x) = P(X \le x) = P(\{\omega \in \Omega : X(\omega) \le x\}).$$

The function $F_X$ is called *the distribution function of* $X$.

The distribution function satisfies the following basic properties :

1. If $x \le y$ then $F(x) \le F(y)$. ($F$ is a non-decreasing function.)
2. If $y \downarrow x$ then $F(y) \downarrow F(x)$. ($F$ is a right-continuous function although it is not necessarily a continuous function.)
3. $\lim_{x \to -\infty} F(x) = 0$; $\lim_{x \to \infty} F(x) = 1$.

**Definition 2.7** A random variable $Y$ is *discrete* if its range is a finite or countably infinite set. That is, there exists a set $S = \{s_1, s_2, \ldots\}$ such that $P(Y \in S) = 1$.

**Definition 2.8** The frequency function of a *discrete* random variable $Y$ is defined by

$$f(y) = P(Y = y).$$

The frequency function of a discrete random variable is known by many other names, such as probability mass function, probability function and density function.

**Definition 2.9.** A random variable $X$ is called *continuous* if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\,du; \quad x \in \mathbb{R},$$

for some integrable function $f : \mathbb{R} \to [0,1]$ called the *probability density function* (pdf) of $X$.

Note : If $f$ is a pdf then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

because $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{x\to\infty} \int_{-\infty}^{x} f(t)\,dt = \lim_{x\to\infty} F(x) = 1.$

In the following, we refer to definitions of expected value and variance which can be found in Hogg, Craig and McKean (2005).

**Definition 2.10.** Let $X$ be a random variable. If $X$ is a continuous random variable with pdf $f(x)$ and

$$\int_{-\infty}^{\infty} |x|\, f(x)\,dx < \infty,$$

then the expectation of $X$ is

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx.$$

If $Y$ is a discrete random variable with pmf $p(y)$ and

$$\sum_{y} |y|\, p(y) < \infty,$$

then the expectation of $Y$ is

$$E[Y] = \sum_{y} y p(y).$$

**Definition 2.11.** Let $X$ be a random variable whose expectation exists. The mean value $\mu$ of $X$ is defind as $\mu = E[X]$.

**Definition 2.12.** Let $X$ be a random variable with finite mean $\mu$ such that $E\left[(X-\mu)^2\right]$ is finite. Then the variance of $X$ is defined as $E\left[(X-\mu)^2\right]$. It is usually denoted by $\sigma^2$ or by $Var(X)$.

Then $Var(X)$ equals

$$\sigma^2 = E\left[(X-\mu)^2\right] = E\left[X^2 - 2\mu X + \mu^2\right];$$

and since $E$ is a linear operator,

$$\begin{aligned}\sigma^2 &= E\left[X^2\right] - 2\mu E\left[X\right] + \mu^2 \\ &= E\left[X^2\right] - 2\mu^2 + \mu^2 \\ &= E\left[X^2\right] - \mu^2.\end{aligned}$$

**Definition 2.13.** (Moment Generating Function (mgf)). Let $X$ be a random variable such that for some $h > 0,$ the expectation of $e^{tX}$ exists for $-h < t < h.$ The generating function of $X$ is defined as function $M(t) = E\left(e^{tX}\right),$ for $-h < t < h.$ We will use the abbreviation mgf to denote moment genrating function of a random variable.

In the following, we recall some distributions of random variables and definitions of mixture models which can be found in Klugman, Panjer and Willmot (2008).

## 2.4 Lognormal Distribution

A random variable $X$ is said to be Lognormally distributed with parameters $\mu$ and $\sigma$ denoted by $X \sim LN(\mu, \sigma),$ if:

CDF : $F_X(x) = \Phi\left(\dfrac{\ln x - \mu}{\sigma}\right); \quad \mu \in R,\ \sigma > 0,\ x > 0.$

PDF : $f_X(x) = \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right),$

Moment: $E\left[X^k\right] = \exp\left(k\mu + \dfrac{1}{2}k^2\sigma^2\right).$

## 2.5 Exponential Distribution

A random variable $X$ is said to be Exponentially distributed with parameter $\theta$ denoted by $X \sim Exp(\theta)$, if:

CDF : $F_X(x) = 1 - e^{\frac{-x}{\theta}}; \quad \theta > 0,\ x > 0.$

PDF : $f_X(x) = \dfrac{e^{\frac{-x}{\theta}}}{\theta},$

Moment: $E\left[X^k\right] = \theta^k \Gamma(k+1),\ k > -1.$

## 2.6 Inverse Exponential Distribution

A random variable $X$ is said to be Inverse Exponentially distributed with parameter $\theta$ denoted by $X \sim IExp(\theta)$, if:

CDF : $F_X(x) = e^{\frac{-x}{\theta}}; \quad \theta > 0,\ x > 0.$

PDF : $f_X(x) = \dfrac{\theta e^{\frac{-\theta}{x}}}{x^2},$

Moment: $E\left[X^k\right] = \theta^k \Gamma(1-k),\ k < 1.$

## 2.7 Inverse Pareto Distribution

A random variable $X$ is said to be Inverse Pareto distributed with parameters $\tau$ and $\theta$ denoted by $X \sim IPa(\tau, \theta)$, if:

CDF : $F_X(x) = \left(\dfrac{x}{x+\theta}\right)^{\tau}; \quad \tau > 0,\ \theta > 0,\ x > 0.$

PDF : $f_X(x) = \dfrac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}}$,

Moment: $E\left[X^k\right] = \dfrac{\theta^k \Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \ -\tau < k < 1.$

$E\left[X^k\right] = \dfrac{\theta^k (-k)!}{(\tau-1)\ldots(\tau+k)}$, if $k$ is negative integer.

## 2.8 Mixture Models

Mixture Models are discrete or continuous weighted combinations of distributions. One motivation for mixing is that the underlying phenomenon may actually be composed of several phenomena that occur with unknown probabilities.

### 2.8.1 The Finite Mixture Models

**Definition 2.14** A random variable $X$ is a $k-point$ *mixture*[1] of the random variables $V_1, V_2, \ldots, V_k$ if its cdf is given by

$$F_X(x) = a_1 F_{V_1}(x) + a_2 F_{V_2}(x) + \ldots + a_k F_{V_K}(x),$$

where all $a_j > 0$ and $a_1 + a_2 + \ldots + a_k = 1.$

**Definition 2.15** A variable component *mixture distribution* has a distribution function that can be written as

$$F(x) = \sum_{j=1}^{K} a_j F_j(x), \quad \sum_{j=1}^{K} a_j = 1, \quad a_j > 0, \quad j = 1, \ldots, K, \quad K = 1, 2, \ldots.$$

[1]The words " Mixed " and " Mixture " have been used interchangeably to refer to the type of distribution described here as well as distributions that are partly discrete and partly continuous.

### 2.8.2 The Infinite Mixture Models

The mixture of distributions is sometimes called compounding. Moreover, it does not need to be restricted to a finite number of distributions.

**Theorem 2.1** Let $X$ have pdf $f_{X|\Lambda}\left(x|\lambda\right)$ and cdf $F_{X|\Lambda}\left(x|\lambda\right)$, where $\lambda$ is a paremeter of $X$. $X$ may have other parameters, however they are not relevant. Let $\lambda$ be a realization of the random variable $\Lambda$ with pdf $f_{\Lambda}\left(\lambda\right)$. Then the unconditional pdf of $X$ is

$$f_X\left(x\right)=\int f_{X|\Lambda}\left(x|\lambda\right) f_{\Lambda}\left(\lambda\right) d\lambda,$$

where the integral has been taken over all values of $\lambda$ with positive probability. The resulting distribution is a mixture distribution. The distribution function can be determined from

$$F_X\left(x\right)=\int_{-\infty}^{x}\int f_{X|\Lambda}\left(y|\lambda\right) f_{\Lambda}\left(\lambda\right) d\lambda dy$$

$$=\int\int_{-\infty}^{x} f_{X|\Lambda}\left(y|\lambda\right) f_{\Lambda}\left(\lambda\right) dy\, d\lambda$$

$$=\int F_{X|\Lambda}\left(x|\lambda\right) f_{\Lambda}\left(\lambda\right) d\lambda.$$

Moments of the mixture distribution can be found from

$$E\left[X^k\right]=E\left[E\left(X^k|\Lambda\right)\right]$$

and, in particular,

$$Var\left(X\right)=E\left[Var\left(X|\Lambda\right)\right]+Var\left[E\left(X|\Lambda\right)\right].$$

## 2.9 Maximum Likelihood Estimation (MLE)

The method of maximum likelihood provides estimators which are usually quite satisfactory and most frequently used in actuarial mathematics.

(Knight, 1999). Suppose that $\mathbf{X}=(X_1,\ldots,X_n)$ are random variables with joint density or frequency function $f(\mathbf{x}:\theta)$ where $\theta\in\Theta.$ Given outcomes $\mathbf{X}=\mathbf{x},$ we define the likelihood function

$$L(\theta)=f(\mathbf{x};\theta);$$

for each possible sample $\mathbf{x}=(x_1,\ldots,x_n)$, the likelihood function $L(\theta)$ is a real-valued function defined on the parameter space $\Theta.$

**Definition 2.16** Suppose that for a sample $\mathbf{x}=(x_1,\ldots,x_n)$, $L(\theta)$ is maximized (over $\Theta$) at $\theta=S(\mathbf{x})$:

$$\sup_{\theta\in\Theta} L(\theta)=L(S(x))$$

(with $S(x)\in\Theta$). Then the statistic $\hat{\theta}=S(\mathbf{X})$ is called the maximum likelihood estimatior (MLE) of $\theta.$

Likelihood equations: If the range of the data does not depend on the data, the parameter space $\theta$ is an open set, and the likelihood function is differentiable with respect to $\boldsymbol{\theta}=(\theta_1,\ldots,\theta_p)$ over $\theta,$ then the maximum likelihood estimate $\hat{\theta}$ satisfies the equations

$$\frac{\partial \ln L(\hat{\boldsymbol{\theta}})}{\partial\theta_k}=o \text{ for } k=1,\ldots,p.$$

These equations are called the likelihood equations and $\ln L(\theta)$ is called the log- likelihood function.

## 2.10 Linear Models

The standard reference for generalized linear models is McCullagh and Nelder (1989).

A vector of observations $\mathbf{x}$ having $n$ components is assumed to be a realization of a random variable $\mathbf{X}$ whose components are independently distributed with means $\boldsymbol{\mu}$. The systematic part of the model is a specification for the vector $\boldsymbol{\mu}$ in terms of a small number of unknow parameters $\beta_1, \ldots, \beta_p$.

In the case of ordinary linear models, this specification takes the form

$$\boldsymbol{\mu} = \sum_{j=1}^{p} \mathbf{z}_j \beta_j , \tag{2.1}$$

where the $\beta s$ are parameters whose values are usually unknown and have to be estimated from the data. If we index the observations by $i,$ then the systematic part of the model may be written

$$E[X_i] = \mu_i = \sum_{j=1}^{p} z_{ij} \beta_j ;\; i = 1, \ldots, n, \tag{2.2}$$

where $z_{ij}$ is the value of the $jth$ covariate for observation $i.$ In matrix notation (where $\boldsymbol{\mu}$ is $n \times 1,$ $\mathbf{z}$ is $n \times p$ and $\boldsymbol{\beta}$ is $p \times 1$) we may write

$$\boldsymbol{\mu} = \mathbf{Z}\boldsymbol{\beta},$$

where $\mathbf{Z}$ is the model matrix and $\boldsymbol{\beta}$ is the vector of parameters.

The components of $\mathbf{X}$ are independent normal variables with constant variance $\sigma^2$ and

$$E[X] = \boldsymbol{\mu} \text{ where } \boldsymbol{\mu} = \mathbf{Z}\boldsymbol{\beta}. \tag{2.3}$$

## 2.11 The Components of a Generalized Linear Model

The Generalized Linear Model is an extension of classical linear models for situations where the response has a non-normal distribution, for example, a Binomial, Poisson, Gamma, inverse Gaussian, Exponential. Thus, a GLM consists of three components:

1. The random component : The distribution of the response variable, $X_i$ (for the *ith* of $n$ independent sample observations) is a member of an exponential family.
2. The systematic component : covariate $z_1, z_2, \ldots, z_p$ produce a linear predictor $\boldsymbol{\eta}$ given by

$$\boldsymbol{\eta} = \sum_{j=1}^{p} z_{ij} \beta_j .$$

3. Link function : The relationship between the random and systematic component. A smooth and invertible linearizing link function $g(\cdot)$, which transforms the expectation of the response variable, $\mu_i = E[X_i]$, to the linear predictor :

$$g(\mu_i) = \boldsymbol{\eta}.$$

The classical linear models have a normal (or Gaussian) distribution in component 1. and the identity function for the link in component 3. and the link function in component 3. may become any monotonic differentiable function.

### 2.11.1 Exponential Family

The theory of generalized linear models is based on a set of probability members of an exponential family. The exponential family can be written in the form

$$f_X\left(x;\theta,\phi\right)=\exp\left\{\frac{\left(d\left(x\right)e\left(\theta\right)-b\left(\theta\right)\right)}{a\left(\phi\right)}+c\left(x,\phi\right)\right\} \tag{2.4}$$

for some specific functions $a(\cdot),b(\cdot)$ and $c(\cdot)$. If $d(x)=x,$ then the function is in canonical form for the random variable $X$. Likewise, if $e(\theta)=\theta,$ it is in canonical form for the parameter $\theta.$ If the substitutions $d(x)=x$ and $e(\theta)=\theta$ are made, the above equation becomes

$$f_X\left(x;\theta,\phi\right)=\exp\left\{\frac{\left(x\theta-b\left(\theta\right)\right)}{a\left(\phi\right)}+c\left(x,\phi\right)\right\}. \tag{2.5}$$

We call $\theta$ the canonical parameter, and $\phi$ the dispersion parameter or scale parameter. If the distribution is parameterized in terms of the mean $\mu$ of $X,$ so that $\theta=g(\mu)$ for some function $g,$ then $g(\mu)$ is the canonical link.

### 2.11.2 Likelihood Functions for Generalized Linear Models

We assume that each component of $\mathbf{X}$ has a distribution in the exponential family, taking the form

$$f_X\left(x;\theta,\phi\right)=\exp\left\{\frac{\left(x\theta-b\left(\theta\right)\right)}{a\left(\phi\right)}+c\left(x,\phi\right)\right\}. \tag{2.6}$$

We write $l(\theta,\phi;x)=\log f_X\left(x;\theta,\phi\right)$ for the log-liklihood function considered as a function of $\theta$ and $\phi,$ $x$ being given. The mean and variance of $X$ can be derived easily from the well known relations.

$$E\left(\frac{\partial l}{\partial \theta}\right)=0 \qquad (2,7)$$

and

$$E\left(\frac{\partial^2 l}{\partial \theta^2}\right)+E\left(\frac{\partial l}{\partial \theta}\right)^2=0. \qquad (2.8)$$

We have from (2.6) that

$$l(\theta;x)=\frac{(x\theta-b(\theta))}{a(\phi)}+c(x,\phi),$$

whence

$$\frac{\partial l}{\partial \theta}=\frac{(x-b'(\theta))}{a(\phi)} \qquad (2.9)$$

and

$$\frac{\partial^2 l}{\partial \theta^2}=\frac{-b''(\theta)}{a(\phi)}, \qquad (2.10)$$

where prime denotes differentiation with respect to $\theta$.

From (2.7) and (2.9) we have

$$0=E\left(\frac{\partial l}{\partial \theta}\right)=\frac{(\mu-b'(\theta))}{a(\phi)},$$

so that

$$E(X)=\mu=b'(\theta).$$

Similarly, from (2.8), (2.9) and (2.10) we have

$$0 = \frac{-b''(\theta)}{a(\phi)} + \frac{\mathrm{var}(X)}{a^2(\phi)},$$

so that

$$\mathrm{var}(X) = b''(\theta)a(\phi).$$

Thus the variance of $\mathbf{X}$ is the product of two functions ; one, $b''(\theta)$, depends on the canonical parameter (and hence on the mean) only and will be called the variance function, while the other is independent of $\theta$ and depends only on $\phi$. The variance function considered as a function of $\mu$ will be written $V(\mu)$.

The function $a(\phi)$ is commonly of the form

$$a(\phi) = \frac{\phi}{\omega},$$

where $\phi$, called the dispersion parameter, is constant over observations, and $\omega$ is a known prior weight that varies from observation to observation.

### 2.11.3 Link Functions

The link function relates the linear predictor $\eta$ to the expected value $\mu$ of a datum $x$. In classical linear models the mean and the linear predictor are identical, and the identity link is plausible in that both $\eta$ and $\mu$ can take any value on the real line. However, when we are dealing with counts and the distribution is Poisson, we must have $\mu > 0$, so that the identity link is less attractive, in part because $\eta$ may be negative while $\mu$ must not be. Models for counts based on independence in crossclassified data lead naturally to multiplicative effects, and this is expressed by the log link, $\eta = \log \mu$, with its inverse $\mu = e^{\eta}$.

For the binomial distribution we have $0 < \mu < 1$ and a link should satisfy the condition that it maps the interval $(0,1)$ on to the whole real line. We shall consider three link functions, namely:

1. Logit

$$\eta = g(\mu) = \log\left(\frac{\mu}{1-\mu}\right);$$

2. Probit

$$\eta = g(\mu) = \Phi^{-1}(\mu);$$

   where $\Phi(\cdot)$ is the Normal cumulative distribution function;

3. Complementary Log-Log

$$\eta = g(\mu) = \log\{-\log(1-\mu)\}.$$

The power family of link is important at least for observations with a positive mean.

This family can be specified either by

$$\eta = \frac{(\mu^{\lambda} - 1)}{\lambda} \tag{2.10a}$$

with the limiting value

$$\eta = \log \mu; \text{ as } \lambda \to 0, \tag{2.10b}$$

or by

$$\eta = \begin{cases} \mu^{\lambda} & ; \quad \lambda \neq 0, \\ \log \mu & ; \quad \lambda = 0. \end{cases}$$

The first form has the advantage of a smooth transition as $\lambda$ passes through zero, but with either form special action has to be take in any computation with $\lambda = 0$.

### 2.11.4 Sufficient Statisfics and Canonical Links

Each of the distributions in the exponential family has a special link function for which there exists a sufficient statistic equal in dimension to $\beta$ in the linear predictor $\boldsymbol{\eta} = \sum \mathbf{z}_{ij}\beta_j$. These canonical links, as they will be called, occur when

$$\theta = \boldsymbol{\eta},$$

where $\theta$ is the canonical parameter as defined in (2.5). The canonical links for distributions in the exponential family are thus:

Normal $\boldsymbol{\eta} = g(\mu) = \mu,$

Poisson $\boldsymbol{\eta} = g(\mu) = \log \mu,$

Gamma $\boldsymbol{\eta} = g(\mu) = \mu^{-1},$

inverse Gaussian $\boldsymbol{\eta} = g(\mu) = \mu^{-2}.$

Note that, if the distribution of the response varies, $X_i$ is a member of exponential family in canonical form then $g(\mu)$ is called the canonical link function.

The next chapter explains how to construct a claim model which will be an infinite mixture distribution that is not a classical distribution.

# CHAPTER III

# CLAIM MODELING

In this chapter, the infinite mixture distributions will be applied to match with two sets of data. The first data sets comprises 99 sample groups generated from a combination of Lognormal, Gamma and Weibull distributions (Stephen and Richard, 2011), in order to simulate insurance data and test work ability of our model. Then our model is applied to the second data sets which consists of 1,296 actual insurance observations.

Considering individual claim policies, let $X_i,\ i = 1, 2, \ldots, n.$ be the Claim Severity of the $i^{th}$ claim. It is assumed that the random variables $X_1, X_2, \ldots, X_n.$ are independent and identically distributed (i.i.d.). Some assumptions and restrictions are specified as follows:

Assumption 1: Claims Severity are non-catastrophic claims.

Assumption 2: No deductible and no reinsurance agreement.

Assumption 3: A recorded Claim Severity is equal to a 1,296 observation

Assumption 4: The claim distributions are skewed to the right.

Assume that the portfolio Claim Severity arises from the 99 sample groups, e.g., combination of claim distributions which are Lognormal, Gamma and Weibull derived by numerical experiments, as listed in subsection 3.3. Moreover, we employ the probability density function (pdf) and the distribution function (df) of claim distribution which are specified in Appendix C to fitting the 99 sample groups.

## 3.1 Classical Distributions

Using the 1,296 observations of motor insurance claims from public non-life insurance companies in Thailand, we fit this data to some classical distributions, (Stephen and Richard, 2011) i.e., Exponential, Inverse Exponential and Lognormal. The maximum likelihood estimation (MLE) is used to estimate the parameters in each distribution.

### 3.1.1 The Model

(1) The probability density function (pdf) for the Exponential distribution is

$$f_X(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right);\ \theta \in R,\ x > 0 \tag{3.1}$$

(2) The pdf for the Inverse Exponential distribution is

$$f_X(x) = \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right);\ \theta > 0,\ x > 0 \tag{3.2}$$

(3) The pdf for the Lognormal distribution is

$$f_X(x) = \frac{1}{\sqrt{2\pi}x\sigma}\exp\left(-\frac{(\ln x - \theta)^2}{2\sigma^2}\right);\ \theta \in R,\ \sigma > 0,\ x > 0. \tag{3.3}$$

### 3.1.2 Estimation for the Model

Let $X_i, i = 1, 2, \ldots, n.$ be the Claim Severity of the $i^{th}$ claim. It is assumed that the random variables $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d.).

Consider the Claim Severity $\{x_i\},(i=1,2,\ldots,n)$, paid for the $i^{th}$ contract. We shall fit the data set $\{x_i\}$ to the Exponential, Inverse Exponential and Lognormal distributions. By MLE, we obtain estimators for the parameters $\theta$ and $\sigma$ as follows:

(1) With the probability density function (pdf) for the Exponential distribution in (3.1), the likelihood function is

$$L(\theta)=\prod_{i=1}^{n}\frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right);\ \theta\in R,\ x>0.$$

Then

$$\ln L(\theta)=\ln\prod_{i=1}^{n}\frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right)$$

$$=\sum_{i=1}^{n}\ln\frac{1}{\theta}\exp\left(-\frac{x_i}{\theta}\right)$$

$$=\ln\left(\frac{1}{\theta^n}\right)+\sum_{i=1}^{n}\left(-\frac{x_i}{\theta}\right)$$

$$=-n\ln\theta-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right);\ \theta\in R,\ x>0.$$

Setting the partial derivatives $\frac{\partial\ln L(\theta)}{\partial\theta}$ to zero, we have

$$\frac{\partial\ln L(\theta)}{\partial\theta}=\frac{-n}{\theta}+\frac{1}{\theta^2}\sum_{i=1}^{n}x_i=0.$$

An estimator $\hat{\theta}$ for the parameter $\theta$ can be obtained by solving the equation $\frac{\partial\ln L(\theta)}{\partial\theta}=0$ where $\hat{\theta}$ is given by: $\hat{\theta}=\frac{\sum_{i=1}^{n}x_i}{n}$.

(2) With the pdf for the Inverse Exponential distribution in (3.2), the likelihood function is

$$L(\theta)=\prod_{i=1}^{n}\frac{\theta}{x_i^2}\exp\left(-\frac{\theta}{x_i}\right);\ \theta>0,\ x>0.$$

Then

$$\begin{aligned}\ln L(\theta) &=\ln\prod_{i=1}^{n}\left[\frac{\theta}{x_i^2}\exp\left(-\frac{\theta}{x_i}\right)\right]\\ &=\sum_{i=1}^{n}\ln\left[\frac{\theta}{x_i^2}\exp\left(-\frac{\theta}{x_i}\right)\right]\\ &=n\ln\theta+\sum_{i=1}^{n}\ln\left[\frac{1}{x_i^2}\exp\left(-\frac{\theta}{x_i}\right)\right]\\ &=n\ln\theta+\sum_{i=1}^{n}\left[-\frac{\theta}{x_i}-\ln x_i^2\right]\\ &=n\ln\theta-\sum_{i=1}^{n}\left[\frac{\theta}{x_i}+2\ln x_i\right];\ \theta>0,\ x>0.\end{aligned}$$

Setting the partial derivatives $\frac{\partial\ln L(\theta)}{\partial\theta}$ to zero, we have

$$\frac{\partial\ln L(\theta)}{\partial\theta}=\frac{1}{\theta}-\sum_{i=1}^{n}\frac{1}{x_i}=0.$$

An estimator $\hat{\theta}$ for the parameter $\theta$ can be obtained by solving the equation $\frac{\partial\ln L(\theta)}{\partial\theta}=0$ where $\hat{\theta}$ is given by: $\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$.

(3) With the pdf for the Lognormal distribution in (3.3), the likelihood function is

$$L(\theta,\sigma)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}x_i\sigma}\exp\left(-\frac{(\ln x_i-\theta)^2}{2\sigma^2}\right);\ \theta\in R,\ \sigma>0,\ x>0.$$

Then

$$\ln L(\theta,\sigma)=\ln\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}x_i\sigma}\exp\left(-\frac{(\ln x_i-\theta)^2}{2\sigma^2}\right)$$

$$=\sum_{i=1}^{n}\ln\frac{1}{\sqrt{2\pi}x_i\sigma}\exp\left(-\frac{(\ln x_i-\theta)^2}{2\sigma^2}\right)$$

$$=-n\ln\sigma-n\ln\sqrt{2\pi}-\sum_{i=1}^{n}\left[\ln x_i+\frac{(\ln x_i-\theta)^2}{2\sigma^2}\right]$$

$$=-n\ln\sigma-\frac{n}{2}\ln 2\pi-\sum_{i=1}^{n}\left[\ln x_i+\frac{1}{2\sigma^2}(\ln x_i-\theta)^2\right]$$

Setting the partial derivatives $\frac{\partial\ln L(\theta)}{\partial\theta}$ and $\frac{\partial\ln L(\theta,\sigma)}{\partial\sigma}$ to zero, we have

$$\frac{\partial\ln L(\theta,\sigma)}{\partial\theta}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\theta)=0$$

$$\frac{\partial\ln L(\theta,\sigma)}{\partial\sigma}=\frac{-n}{\sigma}+\sigma^{-3}\sum_{i=1}^{n}(x_i-\theta)^2=0$$

An estimator $\hat{\theta}$ and $\hat{\sigma}$ for the parameter $\theta$ and $\sigma$ can be obtained by solving these two equations:

$$\frac{\partial\ln L(\theta,\sigma)}{\partial\theta}=0,\ \text{and}\ \frac{\partial\ln L(\theta,\sigma)}{\partial\sigma}=0.$$

The solutions are $\hat{\theta} = \dfrac{\sum_{i=1}^{n} \ln x_i}{n}$, and $\hat{\sigma} = \sqrt{\dfrac{\sum_{i=1}^{n} \left(\ln x_i - \hat{\theta}\right)^2}{n}}$ respectively.

### 3.1.3 Goodness of Fit Test

Goodness of Fit (GOF) test means that one measures the compatibility of a random sample with a theoretical probability distribution function. One GOF test is the K-S test, to decide whether a sample comes from a hypothesized continuous distribution and that based on the Empirical Cumulative Distribution Function (ECDF) which is written to

$$F_n(x) = \frac{1}{n}\left[\textit{Number of observation } \leq x\right].$$

The K-S test is defined by

$$D = \sup_x \left| F_n(x) - F_X^*(x) \right|,$$

where $F_X^*$ is the theoretical cumulative distribution of the distribution being tested.

The K-S test is defined by:

$H_0$ : The data follow a specified distribution.

$H_1$ : The data do not follow a specified distribution.

Level of critical values: The hypothesis regarding the distributional form is rejected at the chosen significance level ($\alpha$) if the test statistic $D$ is greater than the critical value obtained, see Table C.3 in Appendix C.4. Furthermore, we can calculate the $P-$value from the $D-$value and translate the result of the hypothesis test.

Those three classical distributions were applied to the 1,296 observations. An analysis involving some comparisons are presented from the results of the statistical tests.

**Table 3.1** The fitting for classical distributions.

| Distribution | K-S tests | | Estimated Parameter |
|---|---|---|---|
| | $D-$value | $P-$value | |
| Exponential | 0.1961 | $< 0.0100$ | $\hat{\theta} = 1.766 \times 10^4$ |
| Inverse Exponential | 0.0759 | $< 0.0100$ | $\hat{\theta} = 4.190 \times 10^3$ |
| Lognormal | 0.0466 | $< 0.0100$ | $\hat{\sigma} = 1.1804$<br>$\hat{\theta} = 8.9672$ |

Table 3.1 shows the statistical test value for fitting the classical distributions to the 1,296 observations. We found that none of those classical distributions could be fitted to the 1,296 observations at significance level $\alpha = 0.01$, since the $P-$value is less than 0.01. Hence we can reject the null hypothesis and conclude that the data set does not follow these three classical distributions at a 99% confidence level.

Therefore, we selected non-classical distributions which may fit to the 1,296 observations. Next we employed the infinite mixture distribution in 3.2 as a candidate to fit the data with Lognormal, Exponential and Inverse Exponential distributions. Firstly, the K-S test verified that the Lognormal distribution is a better fit than the Exponential and Inverse Exponential distributions. However, the Lognormal distribution cannot derive a cdf. As a result, we have to ignore it. At last, we selected the second best distribution, which is the Inverse Exponential distribution.

## 3.2 Infinite Mixture Model

This section describes the construction of infinite mixture distributions and an estimation of parameters using MLE.

We represent an insurance Claim Severity by the random variable $X$. Let $f_X\left(x|\theta\right)$ denote the pdf of the insurance Claim Severity if the risk parameter is known to be $\theta.$ The heterogeneity in the insurance portfolio is due to variability in the parameter $\theta.$

Let $G\left(\theta\right)=P(\Theta\leq\theta)$ be the cdf of $\Theta,$ where $\Theta$ is the risk parameter viewed as a random variable. $G\left(\theta\right)$ is called the mixing distribution. Let $g\left(\theta\right)$ be the pdf of $\Theta.$ Then

$$h_X\left(x\right) = \int_{R^+} f_X\left(x|\theta\right)g\left(\theta\right)d\theta,\ \forall x\in R^+,$$

is the unconditional pdf of $X$.

### 3.2.1 The Model

An infinite mixture model is composed of Gamma as mixing distribution and Inverse Exponential as mixed distribution. Let $X$ be the Inverse Exponential random variable with parameter $\theta.$ We want to mix an infinite number of Inverse Exponential distributions, each with a different value of $\theta.$ We let the mixing distribution have a pdf of $\theta,$ namely, a Gamma with parameters $\alpha$ and $\beta.$

We begin with to the pdf of the Gamma distribution which is written as

$$g\left(\theta\right) = \frac{\beta^\alpha}{\Gamma\left(\alpha\right)}\theta^{\alpha-1}\exp\left(-\beta\theta\right);\ \alpha,\beta>0,\ \theta>0,$$

and mix it with the pdf of the Inverse Expontial distribution which is written as

$$f_X\left(x|\theta\right) = \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right);\ \theta>0,\ x>0,$$

to obtain the infinite mixture model written as :

$$
\begin{aligned}
h_X(x) &= \int_0^\infty \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\cdot\frac{\beta^\alpha}{\Gamma(\alpha)}\cdot\theta^{\alpha-1}\exp(-\beta\theta)\,d\theta \\
&= \int_0^\infty \frac{\theta^{1+\alpha-1}\beta^\alpha}{x^2\Gamma(\alpha)}\exp\left(-\frac{\theta}{x}-\beta\theta\right)d\theta \\
&= \int_0^\infty \frac{\theta^{(\alpha+1)-1}\left(\frac{1}{x}+\beta\right)^{\alpha+1}}{\Gamma(\alpha+1)}\exp\left[-\theta\left(\frac{1}{x}+\beta\right)\right]d\theta\cdot\frac{\beta^\alpha}{x^2\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{\left(\frac{1}{x}+\beta\right)^{\alpha+1}} \\
&= \frac{\beta^\alpha}{x^2\Gamma(\alpha)}\cdot\frac{\Gamma(\alpha+1)}{\left(\frac{1}{x}+\beta\right)^{\alpha+1}} \\
&= \frac{\alpha\beta^\alpha x^{\alpha-1}}{(1+\beta x)^{\alpha+1}};\ \alpha,\beta>0,\ x>0. \qquad (3.4)
\end{aligned}
$$

Observe that formula (3.4) is similar to the pdf of Inverse Pareto distribution $IPa\left(\alpha,\frac{1}{\beta}\right)$.

In fact, the distribution of $h_X(x)$ is $H(x)=\left(1-\frac{1}{\beta x+1}\right)^\alpha;\ \alpha,\beta>0,\ x>0,$ which is the cdf of the Inverse Pareto distribution $IPa\left(\alpha,\frac{1}{\beta}\right)$. Please see Appendix A for further details.

### 3.2.2 Estimation for the Model

Considering the Claim Severity $x_i$ paid for the $i^{th}$ contract, we fit the $IPa\left(\alpha,\frac{1}{\beta}\right)$ distribution in (3.4) to the 1,296 observations using MLE. The estimated value of parameters $\alpha$ and $\beta$ can be obtained by the following mothod.

Assume that $\mathrm{X} \sim IPa\left(\alpha, \frac{1}{\beta}\right)$ with density

$$h_X(x) = \frac{\alpha\beta^{\alpha}x^{\alpha-1}}{(1+\beta x)^{\alpha+1}}; \quad \alpha, \beta > 0, \; x > 0.$$

The likelihood function can be written as

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha\beta^{\alpha}x^{\alpha-1}}{(1+\beta x)^{\alpha+1}}$$

The log-likelihood function is in the form

$$\begin{aligned}
\ln L(\alpha,\beta) &= \ln \prod_{i=1}^{n} \frac{\alpha\beta^{\alpha}x_i^{\alpha-1}}{(1+\beta x_i)^{\alpha+1}} \\
&= \sum_{i=1}^{n} \ln \frac{\alpha\beta^{\alpha}x_i^{\alpha-1}}{(1+\beta x_i)^{\alpha+1}} \\
&= n\ln\alpha + n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - (\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i).
\end{aligned}$$

Hence, the partial derivatives of the log-likelihood function are

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+\beta x_i),$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}.$$

The two estimations $\hat{\alpha}$ and $\hat{\beta}$ for parameters $\alpha$ and $\beta$ can be obtained by solving these two equations.

$$\frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+\beta x_i) = 0, \tag{3.5}$$

$$\frac{n\alpha}{\beta} - (\alpha + 1)\sum_{i=1}^{n} \frac{x_i}{1+\beta x_i} = 0. \tag{3.6}$$

Because of the difficulty of solving (3.5)-(3.6) algebraically, we preferred to solve the equations numerically by using the Newton-Raphson method to estimate parameters $\alpha$ and $\beta$. We used MATLAB to do this work. These methods are explained in B.2.1 of Appendix B.

## 3.3 The Simulation

We have performed numerical experiments which MATLAB for the 99 sample groups, to fit using infinite mixture distributions.

The 99 sample groups were generated by simulations under the following assumptions.

(1) Sample size

$n$: 200, 400, 600, 800, 1000, 1500, 2000, 4000, 10000, 30000 and 50000

for the groups of two mixed components.

$n$: 150, 450, 600, 750, 1500, 3000, 9000, 12000, 30000, 45000 and 60000

for the groups of three mixed components.

(2) The Simulated data

(2.1) Claim distributions used: Lognormal, Gamma and Weibull.

(2.2) The combination of claim distributions: The $x_i$ is generated based on right skewed distributions according to sample size $n$. We assume that the heterogeneity in the portfolio Claim Severity is due to variability in the parameters and distributions. The group samples are simulated by combination of the claim distributions as shown on Table 3.2.

**Table 3.2** The mixed components.

| Components | Parameters | Distribution |
|---|---|---|
| 2 | Lognormal/ Lognormal<br>Gamma/ Gamma<br>Weibull/ Weibull | Lognormal/Gamma<br>Lognormal/Weibull |
| 3 | Lognormal/ Lognormal/ Lognormal<br>Gamma/ Gamma/ Gamma<br>Weibull/ Weibull/ Weibull | Lognormal/Gamma/Weibull |

Each component mixed has the same number of claims, for details see section C.3 of Appendix C. The simulations comprise 99 groups.

(3) The model of infinite mixture distributions

The model used for fitting to the sample groups is the infinite mixture distribution. A classical distribution (Inverse Exponential distribution) is used as a control to assess the performance of the infinite mixture distributions. To reach the stablity of the results, we ran up to 250 iterations in the simulation.

A flowchart of the claim modelling process, is shown in the Figure 3.1.

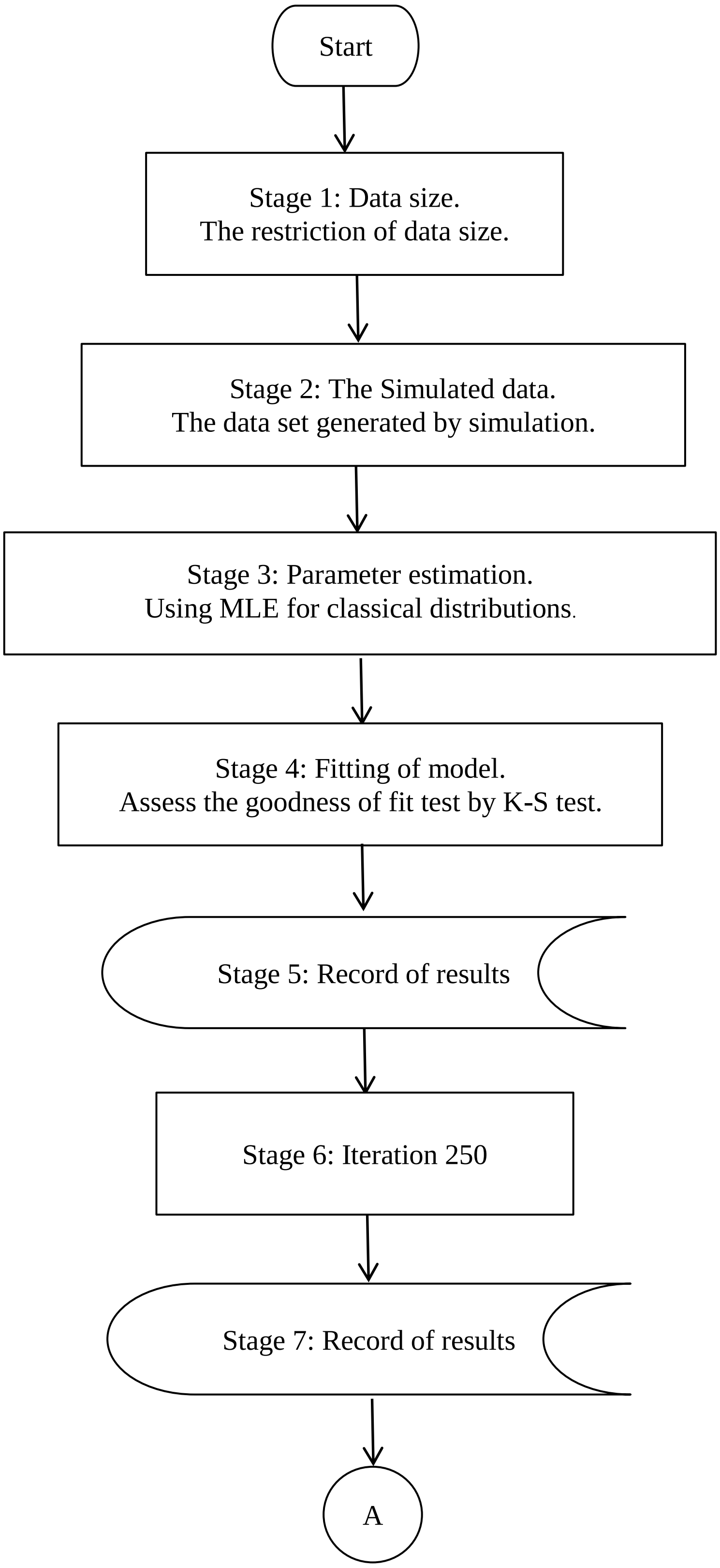


**Figure 3.1** The flowchart of the claim modeling process.

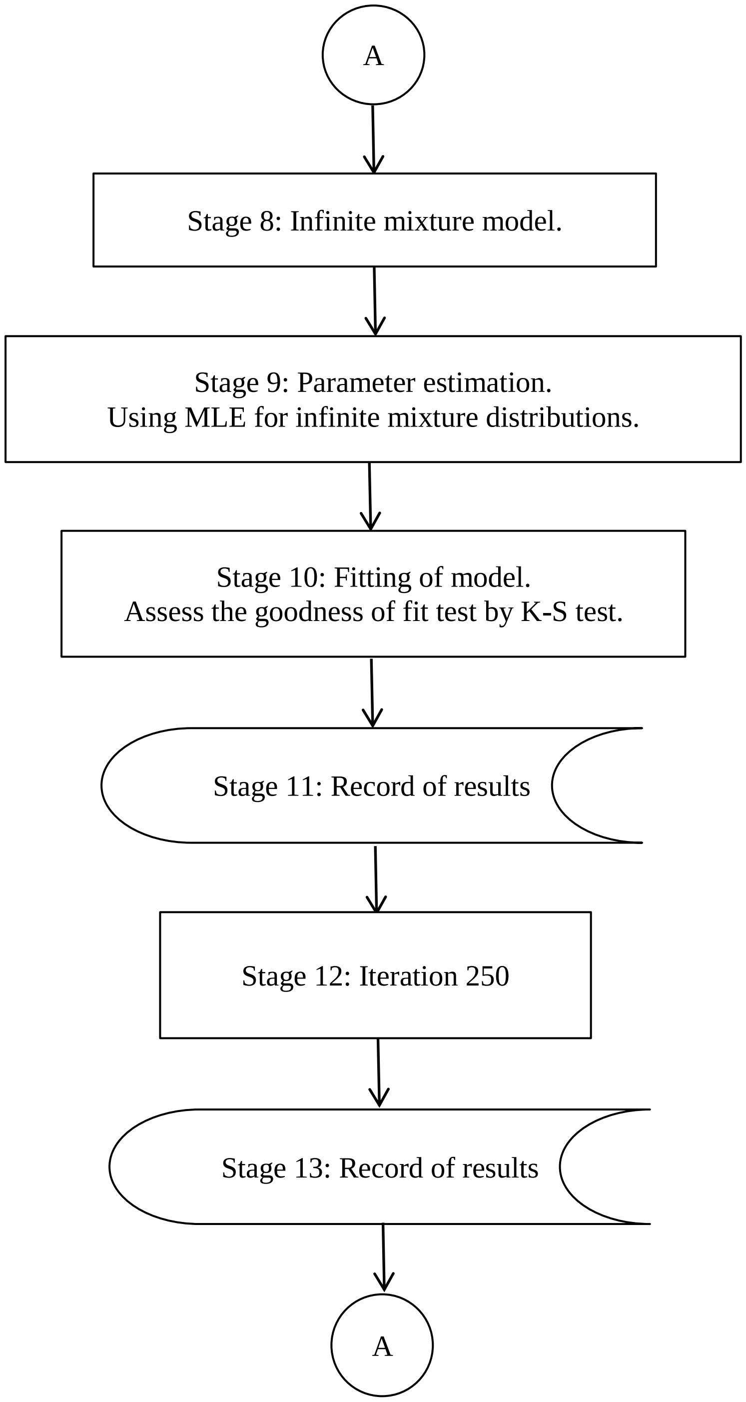


**Figure 3.1** The flowchart of the claim modeling process (Continued).

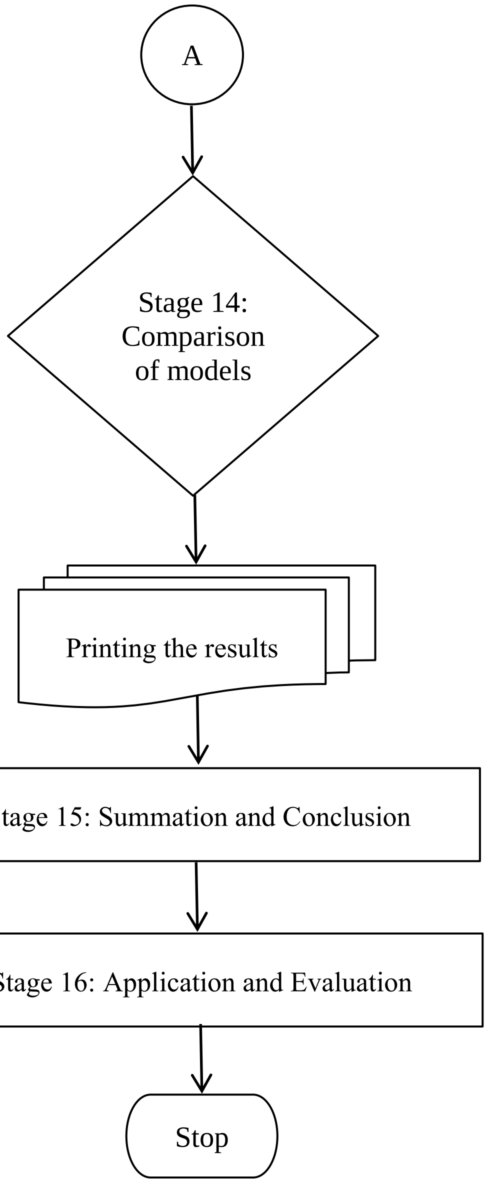


**Figure 3.1** The flowchart of the claim modeling process (Continued).

## 3.4 Simulation Results

The objective of our claim modeling is to assess whether the heterogeneous portfolio Claim Severity data can be fitted to the infinite mixture distributions. The 99 sample groups were simulated by combinations of claim distributions, which are Lognormal, Gamma and Weibull distributions. The parameter estimation was performed by MLE for the classical (Inverse Exponential) and the infinite mixture distributions. We referred K-S test as a statistical test, in which the symbols are defined for explanation the following

$D\_CL$ mean $D-$value of classical distribution

$D\_IF$ means $D-$value of infinite mixture distribution

$P\_CL$ means $P-$value of classical distribution

$P\_IF$ means $P-$value of infinite mixture distribution

We present the value of $D\_CL$, $D\_IF$, $P\_CL$ and $P\_IF$ in tables. The results are shown in the following tables.

Tables 3.3-3.6 show the values of $D\_CL$, $D\_IF$, $P\_CL$ and $P\_IF$ for each sample size. The results are that the infinite mixture distribution can be suitable for data mixed of Lognormal and Lognormal when $n = 200, 400, 600, 800, 1000$. We found that for 94 groups the infinite mixture has a $D-$value in the K-S test which is less than with the classical distributions. Although the infinite mixture may not be suitable for the data sets, it can modified so that it sits better than some classical distribution.

**Table 3.3** The fitting distribution to 2 mixed components (parameters).

| 2 mixed components | n | D_CL | D_IF | P_CL | P_IF |
|---|---|---|---|---|---|
| Lognormal/Lognormal | 200 | 0.4542 | 0.0575 | <0.01 | >0.20 |
| | 400 | 0.4490 | 0.0548 | <0.01 | >0.15 |
| | 600 | 0.4451 | 0.0501 | <0.01 | >0.05 |
| | 800 | 0.4494 | 0.0485 | <0.01 | >0.05 |
| | 1000 | 0.4484 | 0.0475 | <0.01 | >0.01 |
| | 1500 | 0.4490 | 0.0457 | <0.01 | <0.01 |
| | 2000 | 0.4529 | 0.0446 | <0.01 | <0.01 |
| | 4000 | 0.4538 | 0.0434 | <0.01 | <0.01 |
| | 10000 | 0.4505 | 0.0423 | <0.01 | <0.01 |
| | 30000 | 0.4511 | 0.0413 | <0.01 | <0.01 |
| | 50000 | 0.4512 | 0.0413 | <0.01 | <0.01 |
| Gamma/Gamma | 200 | 0.4405 | 0.3294 | <0.01 | <0.01 |
| | 400 | 0.4674 | 0.3266 | <0.01 | <0.01 |
| | 600 | 0.4673 | 0.3259 | <0.01 | <0.01 |
| | 800 | 0.4673 | 0.3256 | <0.01 | <0.01 |
| | 1000 | 0.4673 | 0.3257 | <0.01 | <0.01 |
| | 1500 | 0.4673 | 0.3253 | <0.01 | <0.01 |
| | 2000 | 0.4673 | 0.3253 | <0.01 | <0.01 |
| | 4000 | 0.4672 | 0.3253 | <0.01 | <0.01 |
| | 10000 | 0.4672 | 0.3251 | <0.01 | <0.01 |
| | 30000 | 0.4672 | 0.3250 | <0.01 | <0.01 |
| | 50000 | 0.4672 | 0.3250 | <0.01 | <0.01 |
| Weibull/Weibull | 200 | 0.3859 | 0.1999 | <0.01 | <0.01 |
| | 400 | 0.3865 | 0.1950 | <0.01 | <0.01 |
| | 600 | 0.3866 | 0.1953 | <0.01 | <0.01 |
| | 800 | 0.3823 | 0.1956 | <0.01 | <0.01 |
| | 1000 | 0.3795 | 0.1969 | <0.01 | <0.01 |
| | 1500 | 0.3789 | 0.1900 | <0.01 | <0.01 |
| | 2000 | 0.3834 | 0.1934 | <0.01 | <0.01 |
| | 4000 | 0.3784 | 0.1929 | <0.01 | <0.01 |
| | 10000 | 0.3802 | 0.1919 | <0.01 | <0.01 |
| | 30000 | 0.3816 | 0.1904 | <0.01 | <0.01 |
| | 50000 | 0.3811 | 0.1905 | <0.01 | <0.01 |

**Table 3.4** The fitting distribution to 2 mixed components.

| 2 mixed components | n | D_CL | D_IF | P_CL | P_IF |
|---|---|---|---|---|---|
| Lognormal/Gamma | 200 | 0.5588 | 0.2937 | <0.01 | <0.01 |
| | 400 | 0.5759 | 0.2941 | <0.01 | <0.01 |
| | 600 | 0.5448 | 0.2944 | <0.01 | <0.01 |
| | 800 | 0.5578 | 0.2943 | <0.01 | <0.01 |
| | 1000 | 0.5597 | 0.2944 | <0.01 | <0.01 |
| | 1500 | 0.5605 | 0.2942 | <0.01 | <0.01 |
| | 2000 | 0.5546 | 0.2940 | <0.01 | <0.01 |
| | 4000 | 0.5674 | 0.2941 | <0.01 | <0.01 |
| | 10000 | 0.5570 | 0.2941 | <0.01 | <0.01 |
| | 30000 | 0.5577 | 0.2941 | <0.01 | <0.01 |
| | 50000 | 0.5600 | 0.2942 | <0.01 | <0.01 |
| Lognormal/Weibull | 200 | 0.4649 | 0.1884 | <0.01 | <0.01 |
| | 400 | 0.4664 | 0.1900 | <0.01 | <0.01 |
| | 600 | 0.4623 | 0.1763 | <0.01 | <0.01 |
| | 800 | 0.4635 | 0.1754 | <0.01 | <0.01 |
| | 1000 | 0.4617 | 0.1758 | <0.01 | <0.01 |
| | 1500 | 0.4608 | 0.1787 | <0.01 | <0.01 |
| | 2000 | 0.4603 | 0.1767 | <0.01 | <0.01 |
| | 4000 | 0.4606 | 0.1747 | <0.01 | <0.01 |
| | 10000 | 0.4614 | 0.1745 | <0.01 | <0.01 |
| | 30000 | 0.4608 | 0.1739 | <0.01 | <0.01 |
| | 50000 | 0.4609 | 0.1739 | <0.01 | <0.01 |

**Table 3.5** The fitting distribution to 3 mixed components (parameters).

| 3 mixed components | n | D_CL | D_IF | P_CL | P_IF |
|---|---|---|---|---|---|
| Lognormal/Lognormal/ Lognormal | 150 | 0.3723 | 0.1272 | <0.01 | <0.01 |
| | 450 | 0.3322 | 0.1143 | <0.01 | <0.01 |
| | 600 | 0.3369 | 0.1193 | <0.01 | <0.01 |
| | 750 | 0.3647 | 0.1122 | <0.01 | <0.01 |
| | 1500 | 0.3415 | 0.1182 | <0.01 | <0.01 |
| | 3000 | 0.3424 | 0.1167 | <0.01 | <0.01 |
| | 9000 | 0.3561 | 0.1144 | <0.01 | <0.01 |
| | 12000 | 0.3556 | 0.1141 | <0.01 | <0.01 |
| | 30000 | 0.3634 | 0.1136 | <0.01 | <0.01 |
| | 45000 | 0.3465 | 0.1134 | <0.01 | <0.01 |
| | 60000 | 0.3427 | 0.1130 | <0.01 | <0.01 |
| Gamma/Gamma/Gamma | 150 | 0.4724 | 0.2388 | <0.01 | <0.01 |
| | 450 | 0.4723 | 0.2386 | <0.01 | <0.01 |
| | 600 | 0.4723 | 0.2386 | <0.01 | <0.01 |
| | 750 | 0.4723 | 0.2386 | <0.01 | <0.01 |
| | 1500 | 0.4722 | 0.2383 | <0.01 | <0.01 |
| | 3000 | 0.4722 | 0.2383 | <0.01 | <0.01 |
| | 9000 | 0.4722 | 0.2382 | <0.01 | <0.01 |
| | 12000 | 0.4722 | 0.2382 | <0.01 | <0.01 |
| | 30000 | 0.4721 | 0.2382 | <0.01 | <0.01 |
| | 45000 | 0.4721 | 0.2382 | <0.01 | <0.01 |
| | 60000 | 0.4721 | 0.2382 | <0.01 | <0.01 |
| Weibull/Weibull/Weibull | 150 | 0.4350 | 0.1548 | <0.01 | <0.01 |
| | 450 | 0.4350 | 0.1548 | <0.01 | <0.01 |
| | 600 | 0.4363 | 0.1508 | <0.01 | <0.01 |
| | 750 | 0.4393 | 0.1531 | <0.01 | <0.01 |
| | 1500 | 0.4382 | 0.1529 | <0.01 | <0.01 |
| | 3000 | 0.4392 | 0.1517 | <0.01 | <0.01 |
| | 9000 | 0.4361 | 0.1503 | <0.01 | <0.01 |
| | 12000 | 0.4383 | 0.1506 | <0.01 | <0.01 |
| | 30000 | 0.4368 | 0.1507 | <0.01 | <0.01 |
| | 45000 | 0.4367 | 0.1506 | <0.01 | <0.01 |
| | 60000 | 0.4367 | 0.1504 | <0.01 | <0.01 |

**Table 3.6** The fitting distribution to 3 mixed components.

| 3 mixed components | n | D_CL | D_IF | P_CL | P_IF |
|---|---|---|---|---|---|
| Lognormal/Gamma/ | 150 | 0.3945 | 0.2357 | < 0.01 | < 0.01 |
| Weibull | 450 | 0.4177 | 0.2378 | < 0.01 | < 0.01 |
| | 600 | 0.3861 | 0.2382 | < 0.01 | < 0.01 |
| | 750 | 0.4301 | 0.2378 | < 0.01 | < 0.01 |
| | 1500 | 0.4300 | 0.2375 | < 0.01 | < 0.01 |
| | 3000 | 0.4241 | 0.2341 | < 0.01 | < 0.01 |
| | 9000 | 0.4329 | 0.2354 | < 0.01 | < 0.01 |
| | 12000 | 0.4322 | 0.2335 | < 0.01 | < 0.01 |
| | 30000 | 0.4465 | 0.2342 | < 0.01 | < 0.01 |
| | 45000 | 0.4471 | 0.2337 | < 0.01 | < 0.01 |
| | 60000 | 0.4435 | 0.2335 | < 0.01 | < 0.01 |

## 3.5 An Application

Rehearsing to fit the 1,296 observations with the Inverse Pareto distribution $IPa\left(\alpha, \frac{1}{\beta}\right)$, we used the K-S test for testing of model fitting. The histogram for the observations in log scale is illustrated in Figure 3.2.

Table 3.7 shows the statistical test value for fitting of the Inverse Pareto distribution and the estimated parameters. The results of the K-S test reveal a $P-$ value for Inverse Pareto distribution of 0.0482 which is greater than 0.01. Hence, we can conclude that the 1,296 observations can be fitted by the Inverse Pareto distribution with a 99% confidence level. The estimated parameters for the Inverse Pareto distribution are $\hat{\alpha} = 4.7260$ and $\hat{\beta} = 8.7870 \times 10^{-4}$.

**Table 3.7** The fitting of Infinite Mixture distribution.

| Distribution | K-S test | | Estimated Parameter |
|---|---|---|---|
| | $D-$value | $P-$value | |
| Inverse Pareto | 0.0381 | 0.0482 | $\hat{\alpha} = 4.7260$<br>$\hat{\beta} = 8.7870 \times 10^{-4}$ |

In Figure 3.3, the solid line shows the Empirical Cumulative Distribution Function (ECDF) while the dashed line is the cdf of the Exponential distribution.

In Figure 3.4, the solid line shows the ECDF while the dashed line is the cdf of the Inverse Exponential distribution.

In Figure 3.5, the solid line shows the ECDF while the dashed line is the cdf of the Lognormal distribution.

In Figure 3.6, the solid line shows the ECDF while the dashed line is the cdf of the Inverse Pareto distribution.

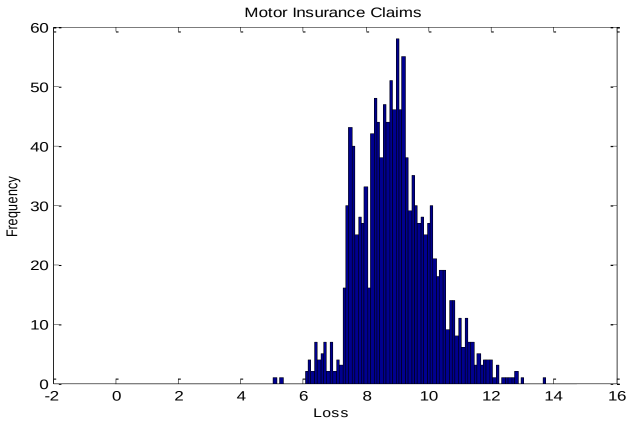


**Figure 3.2** Histogram (log scale).

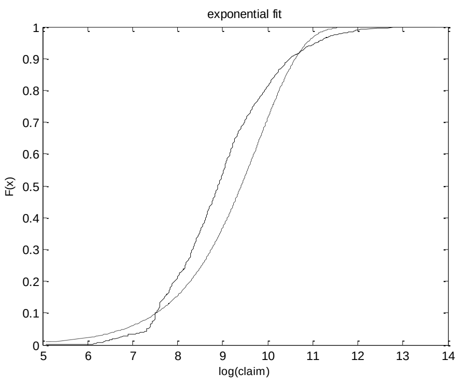


**Figure 3.3** Model versus data cdf plot for the claim data set.

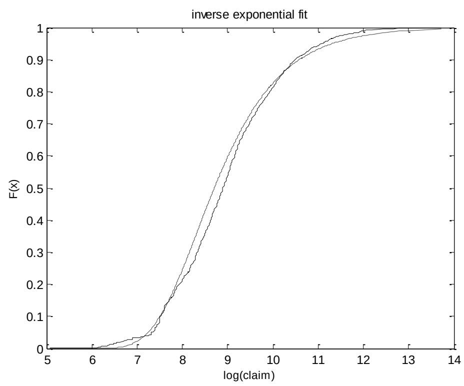


**Figure 3.4** Model versus data cdf plot for the claim data set.

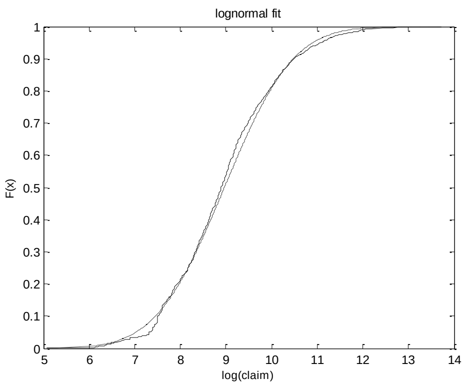


**Figure 3.5** Model versus data cdf plot for the claim data set.

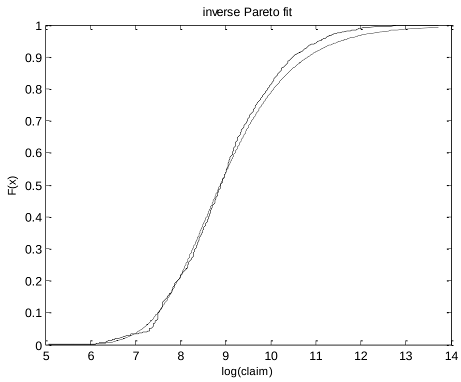


**Figure 3.6** Model versus data cdf plot for the claim data set.

Figures 3.7, 3.8, 3.9 and 3.10 showed the P-P plot for Exponential, Inverse Exponential, Lognormal and Inverse Pareto distributions, respectively.

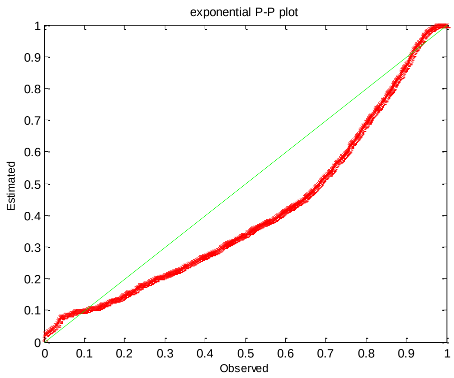


**Figure 3.7** P-P plot for Exponential Distribution.

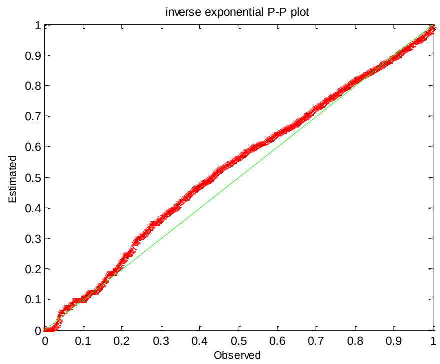


**Figure 3.8** P-P plot for Inverse Exponential distributon.

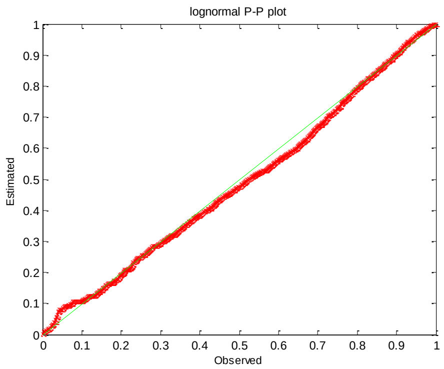


**Figure 3.9** P-P plot for Lognormal distribution.

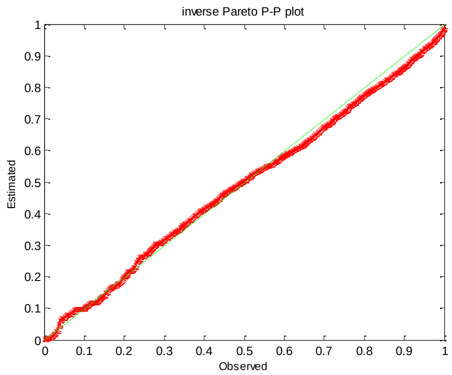


**Figure 3.10** P-P plot for Inverse Pareto distribution.

This chapter has described the construction of the infinite mixture model, and the next chapter is to use the results of this model to price an insurance premium.

# CHAPTER IV

# INSURANCE PRICING

Premium determination is a primary task in the insurance industry:  to conduct business and also to make it competitive in the market. Pricing of the insurance is based on risk factors such as driver's age, gender, marital status, type of car driven or vehicle age which involves constitutional rights and actuarial fairness.

In practice, the linear models are often inadequate because response variables rely on normal distribution. The Claim Severity or loss distributions are defined on the positive real line, especially the fat-tailed and skewed right distribution, whereas the Claim Frequency based on a discrete distribution is a natural approach for counting data and making non-negative observations, (Tes,  2009). Referring to Ohlsson and Johansson (2010), by far the most practical solution to linearise the non-life motor insurance is the generalized linear model (GLM). In other words, the GLM drops that restriction and provides a more suitable solution to this problem. The GLM is an extension which allows the model to follow the distribution, rather than other normal distributions.

Pure Premium can determine from two components which are frequency and severity distributions of the potential claims. To price the insurance premium, it is necessary to take the mean of the frequency and the severity distribution which produces the pure premium:

$$Frequency \times Severity = Pure\,Premium$$

In insurance premium pricing, the GLM is often used to estimate premiums for different individual characteristics of the insured person, including the characteristics of the car.

In this study, data consists of the Claim Severity for each policy and several characteristics of the insured person, such as age and gender. Each policy is assumed to have only one claim. Therefore, the expected value of Claim Frequency equals 1.

The aim of this research is to solve the problem of insurance pricing of motor insurance claims using the observations from the public non-life insurance companies in Thailand, where the data of Claim Severity are modeled by an Inverse Pareto distribution. We employed GLM and mainly focused on the types of a) age and b) gender, which are the two major rating factors.

Our work in this section is to organize as follows: Section 4.1 presents the Testing of Data beginning with testing normality in 4.1.1, nonlinearity in 4.1.2 and introduction of nonlinear model Generalized Linear Model (GLM) in 4.1.3. In Section 4.2, we refer to the concept of GLM employing Inverse Pareto in 4.2.1. In the next section 4.2.2, we present the estimation of GLM employed Inverse Pareto, followed by and its corresponding results in section 4.3. In other word, these sections using the materials and methods for calculating the predicted values of Claim Severity, since it shows the construction of a GLM where the response is modeled by Inverse Pareto distribution. Moreover, a comparison of the results from all the factors concerned is also presented in 4.3. Finally, pricing of the insurance premium for different individual characteristics of the insured person are presented in the Application for Prediction of the Insurance Premium in section 4.4.

## 4.1 The Test of Data

We classify the data set by testing normality, nonlinearity and then explain the Generalized Linear Model (GLM) in Section 4.1.1-4.1.3.

### 4.1.1 Test for Normality

Test for Normality by using the Shapiro Wilk Test, and found that p-Value < e-16 < 0.05. Thus, the distribution is not a normality.

### 4.1.2 Test for Nonlinearity

(1) Test for Unit root using the following:

(a) Augmented Dickey-Fuller Test, and found that p-Value = 0.01 < 0.05. Hence, the distribution is a nonlinearity.

(b) Phillips-Perron Test, and found that p-Value = 0.01 < 0.05. Hence, the distribution is a nonlinearity.

(2) Test for trend stationarity using Kwiatkowsk – Phillips – Schmidt – Shin (KPSS) Test, and found that p-Value = 0.01 > 0.05 Hence, the distribution is a nonlinearity.

We will use Program R for test Normality and Nonlinearity (see Appendix D).

### 4.1.3 The Generalized Linear Model (GLM)

The objective of both linear models and GLM are to express the relationship between an observed response variable, $X$, and a number of covariates, $z$. Both models view the observations $x_1, x_2, \ldots, x_n$ as realizations of the random variables

$X_1, X_2, \ldots, X_n$. Thus, $\mathbf{X}$ represents a vector of the random variables $X_1, \ldots, X_n$.

Whose observations are of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$.

For the classical linear model in the form, the components of $X$ have independent normal distributions with constant variance $\sigma^2$ and

$$E[X_i] = \mu_i \quad \text{where} \quad \mu_i = \sum_{j=1}^{p} z_{ij}\beta_j \,;\; i = 1, \ldots, n.$$

GLM is the extended version of linear model. It allows the population means depend on a linear predictor via a nonlinear link function, transforming between response and covariate variables.

The goal of building a successful model, however, lies in selecting the suitable link function to use.

For example

Assume that $X_i \sim Poisson(\mu_i)$.

so that $g(\mu_i) = \log(\mu_i) = \sum_{j}^{p} z_{ij}\beta_j$ is canonical link.

The mean of $X_i$, is $\mu_i = \exp\left(\sum_{j}^{p} z_{ij}\beta_j\right)$. This will ensure that all of the predicted values are positive. The log link is the most suitable link function to use. The canonical links often have good properties, so the choosing of the link function should be based on prior expectation. Alternativly, some other software packages used to predict the values are availabe, i.e., GLIM, R, S-PLUS, SAS, Stata, Genata, SYSTAT, etc. However, those software are not suited to the simulation of the Inverse Pareto distribution.

As previously mentioned, the predicted values by the GLM technique provide a response random variable, $X$, that has a distribution in the exponential family.

In connecting to this work from Chapter III, we refer to the claim data which has been fitted with Inverse Pareto distribution. Finally, we define a flowchart of the premium calculation that leads to pricing the insurance premium.

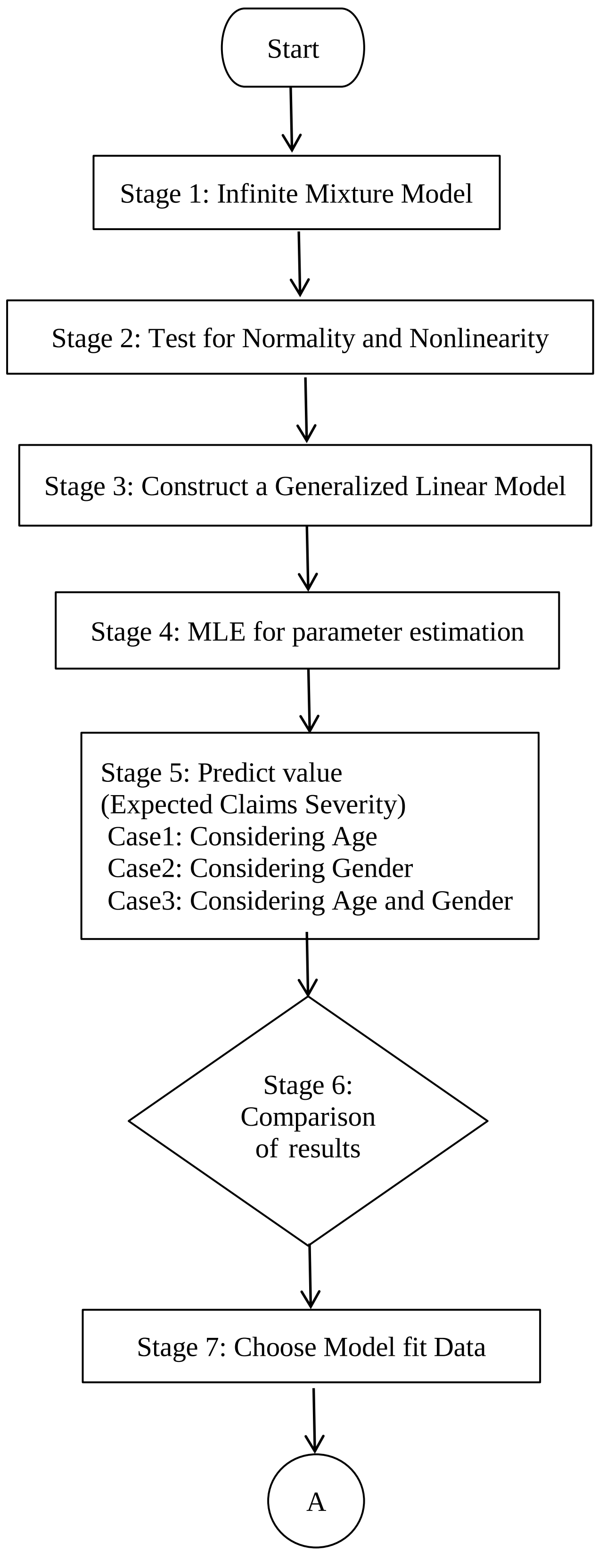


**Figure 4.1** The flowchart of the premium calculation.

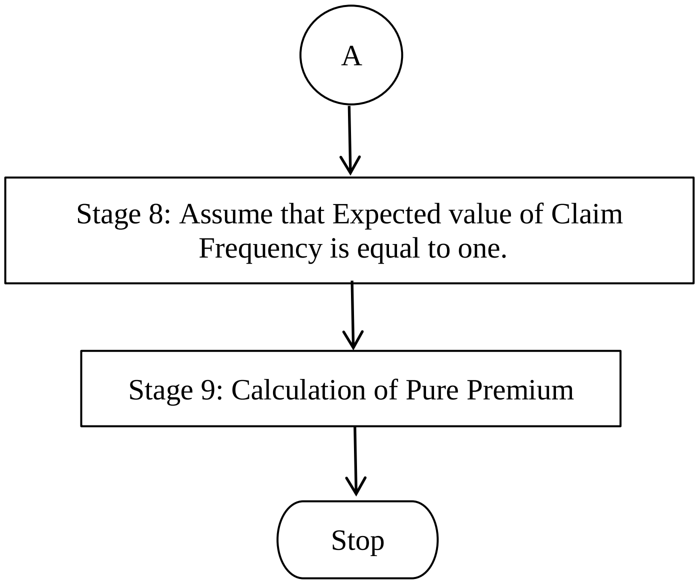


**Figure 4.1** The flowchart of the premium calculation (Continued).

## 4.2 Construction of a Generalized Linear Model

We explain the Inverse Pareto model in 4.2.1 and the Estimation for the model in 4.2.2.

**Definition 4.1** (Klugman *et al.*, 2008) Suppose a parametric distribution has parameters $\mu$ and $\theta$, where $\mu$ is the mean and $\theta$ is a vector of additional parameters. Let its cdf be $F\left(x|\mu,\theta\right)$. The mean must not depend on the additional parameters and the additional parameters must not depend on the mean. Let $\mathbf{z}$ be a vector of covariates for an individual, Let $\beta$ be a vector of coefficients, and let $\eta(\mu)$ and $c(y)$ be functions. The generalized linear model then states that the random variable, $X$, has as its distribution functions.

$$F\left(x|\mathbf{z},\theta\right)=F\left(x|\mu,\theta\right),$$

where $\mu$ is such that $\eta(\mu)=c\left(\beta^T\mathbf{z}\right)$.

Let $\mathbf{z}=(z_1,\ldots,z_n)^T$ be the column vector of the $z$ values and $\beta=\left(\beta_1,\ldots,\beta_p\right)^T$ the column vector of coefficients.

### 4.2.1 The Inverse Pareto Model

Assume that $X\sim Inverse\,Pareto\left(\alpha,\dfrac{1}{\beta}\right)$, abbreviated to $X\sim IPa\left(\alpha,\dfrac{1}{\beta}\right)$, with density

$$h_X\left(x|\alpha,\beta\right)=\frac{\alpha\beta^{\alpha}x^{\alpha-1}}{\left(1+\beta x\right)^{\alpha+1}};\quad \alpha,\beta>0,x>0$$

and distribution function (cdf)

$$H_X\left(x|\alpha,\beta\right)=\left(1-\frac{1}{\beta x+1}\right)^{\alpha}.$$

Consider the moment of $X$,

$$E\left[X^k\right]=\frac{\Gamma\left(\alpha+k\right)\Gamma\left(1-k\right)}{\beta^k\Gamma\left(\alpha\right)},\quad -\alpha<k<1.$$

Approximate $E\left[X\right]$ with $E\left[X^k\right]$when for $k$ close to 1. $E\left[X\right]$ does not exist.

We estimate the value of $E\left[X^k\right]$ when $k$ is equal to $0.1, 0.2, \ldots, 0.9, 0.91, \ldots, 0.99$.

We show $k=0.95$.

$$E\left[X^{0.95}\right]=\frac{\Gamma\left(\alpha+0.95\right)\Gamma\left(1-0.95\right)}{\beta^{0.95}\Gamma\left(\alpha\right)},$$

Approximate $\Gamma\left(\alpha+0.95\right)$ by $\Gamma\left(\alpha+1\right)$ when $\alpha\in\left[0.001, 2\right]$. Therefore, the approximation error is not over $0.10$.

$$=\frac{\Gamma\left(\alpha+1\right)\Gamma\left(0.05\right)}{\beta^{0.95}\Gamma\left(\alpha\right)}=\frac{\alpha\,\Gamma\left(0.05\right)}{\beta^{0.95}}$$

Thus, $E\left[X^{0.95}\right]=\frac{\alpha\Gamma\left(0.05\right)}{\beta^{0.95}}$.

In this thesis, we assume that mean or expected value of an Inverse Pareto distributed random variable X with parameters $\alpha$ and $\frac{1}{\beta}$ are given by $\frac{\alpha\Gamma\left(0.05\right)}{\beta^{0.95}}$.

Next, we construct a GLM for the observations for some non-life insurance public companies in Thailand when the Claim Severity is modeled on Inverse Pareto distribution.

None of the two parameters $\alpha$ and $\frac{1}{\beta}$ in the Inverse Pareto distribution reflect the mean. To make the mean one of the parameters, we can set $\mu = \frac{\alpha\Gamma(0.05)}{\beta^{0.95}}$ or, equivalently, replacing $\alpha$ with $\frac{\mu\beta^{0.95}}{\Gamma(0.05)}$.

The cdf is now

$$H\left(x|\mu,\beta\right) = \left(1 - \frac{1}{\beta x + 1}\right)^{\frac{\mu\beta^{0.95}}{\Gamma(0.05)}}$$

and the pdf is

$$h\left(x|\mu,\beta\right) = \frac{\frac{\mu\beta^{0.95}}{\Gamma(0.05)} \cdot \beta^{\frac{\mu\beta^{0.95}}{\Gamma(0.05)}} \cdot x^{\left[\frac{\mu\beta^{0.95}}{\Gamma(0.05)} - 1\right]}}{\left(1+\beta x\right)^{\left[\frac{\mu\beta^{0.95}}{\Gamma(0.05)} + 1\right]}}.$$

By definition, one may link the covariates to the mean by using $n(\mu) = \mu$ and $c\left(\beta^T z\right) = \exp\left(\beta^T z\right)$. Setting $n(\mu) = c\left(\beta^T z\right)$ , then $\mu = \exp\left(\beta^T z\right)$.

Note that it is expected that all of the predicted values are positive.

For each observation, the Inverse Pareto distribution uses the parameter $\beta$ directly, while the parameter $\alpha$ is derived from the value of $\beta$ and the covariates for that observation.

Our interest lies in investigating the risk factors that affect the Claim Severity for each policy and specifically the risk factors that correspond to the insured person.

The data consist of the Claim Severity for each policy which we want to predict and several characteristics of the driver are based on two rating variables: age of driver and gender of driver.

*(a)* Let $X_1, \ldots, X_n$ be the Claim Severities of $n$ independent claims. These are considered to be *random quantities.*

*(b)* Let $z_{i2}$ be the age of the driver and let and $z_{i3}$ be the gender of driver (e.g. the $i^{th}$ element of $z_{i3}$ is 1 when the $i^{th}$ observation is women, and $0$ if men). These are considered as *fixed quantities.*

The matrix notation presents as follows:

(a) Let $\mathbf{X}$ be the $n$ dimensional column vector of response variables; $\mathbf{X} = \left(X_1, \ldots, X_n\right)^T$.

(b) Let $\beta$ be the $p$ dimensional column vector of coefficients; $\beta = \left(\beta_1, \ldots, \beta_p\right)^T$.

(c) Let $\mathbf{z}$ be the $p$ dimensional column vector of covariates; $\mathbf{z} = \left(z_1, \ldots, z_p\right)^T$.

The design matrix is

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & & z_{2p} \\ \vdots & & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{np} \end{bmatrix}$$

If $z_{i1} \equiv 1$ then $\beta_1$ is *intercept* of the model.

Under the GLM, the mean of $X_i$ is

$$E\left[X_i\right] \equiv \mu_i = \exp\left\{\sum_{j=1}^{p} \beta_j^T z_{ij}\right\}.$$

We are interested in investigating factors that affect the Claim Severity for each policy by considering separation in 3 cases as follows:

**Case 1:** considering the age of the driver $\left(z_{i2}\right)$ by substitution $\mu_i = \exp\left(\beta_1 z_{i1} + \beta_2 z_{i2}\right)$

**Case 2:** considering the gender of the driver $\left(z_{i3}\right)$ by substitution $\mu_i = \exp\left(\beta_1 z_{i1} + \beta_3 z_{i3}\right)$

**Case 3:** considering the age $\left(z_{i2}\right)$ and gender $\left(z_{i3}\right)$ of the driver by substitution $\mu_i = \exp\left(\beta_1 z_{i1} + \beta_2 z_{i2} + \beta_3 z_{i3}\right)$

### 4.2.2 Estimation for the Inverse Pareto Model

Considering the amount $\{x_i\},(i=1,2,\ldots,n),$ paid for the $i^{th}$ contract. We shall fit the data set $\{x_i\}$ to the Inverse Pareto distributions. By MLE, we obtain estimations for parameter $\beta,\beta_1,\beta_2$ and $\beta_3$ as follows:

The pdf for the Inverse Pareto distribution is

$$h_X\left(x|\mu,\beta\right) = \frac{\frac{\mu\beta^{0.95}}{\Gamma(0.05)} \cdot \beta^{\frac{\mu\beta^{0.95}}{\Gamma(0.05)}} \cdot x^{\left[\frac{\mu\beta^{0.95}}{\Gamma(0.05)}-1\right]}}{\left(1+\beta x\right)^{\left(\frac{\mu\beta^{0.95}}{\Gamma(0.05)}+1\right)}}$$

Its likelihood function can be written as

$$L\left(x|\mu,\beta\right) = \prod_{i=1}^{n} \frac{\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)} \beta^{\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}} x_i^{\left(\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}-1\right)}}{\left(1+\beta x_i\right)^{\left[\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}+1\right]}}$$

The log-likelihood function is in the from

$$\ln L\left(x|\mu,\beta\right) = \sum_{i=1}^{n} \ln\left(\frac{\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)} \beta^{\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}} x_i^{\left(\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}-1\right)}}{\left(1+\beta x_i\right)^{\left[\frac{\mu_i\beta^{0.95}}{\Gamma(0.05)}+1\right]}}\right) \qquad \ldots(4.1)$$

**Case1**: By substitution $\mu_i = \exp\left(\beta_1 z_{i1} + \beta_2 z_{i2}\right)$ in (4.1)

$$\ln L\left(x\middle|\mu,\beta\right)=\sum_{i=1}^{n}\ln\left(\frac{\dfrac{e^{\left(\beta_1 z_{i1}+\beta_2 z_{i2}\right)}\beta^{0.95}}{\Gamma\left(0.05\right)}\beta^{\frac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{0.95}}{\Gamma(0.05)}}x_i^{\left(\frac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}}{\Gamma(0.05)}-1\right)}}{\left(1+\beta x_i\right)^{\left[\frac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{0.95}}{\Gamma(0.05)}+1\right]}}\right)$$

Hence, the partial derivatives of the log-likelihood function are

$$\frac{\partial \ln L(x|\mu,\beta)}{\partial\beta}=0.95\frac{n}{\beta}+$$

$$\frac{1}{\Gamma\left(0.05\right)}\sum_{i=1}^{n}\left[\begin{array}{l} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln\left(1+\beta x_i\right) \\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i-\dfrac{\Gamma\left(0.05\right)x_i}{1+\beta x_i} \end{array}\right]$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l} \left(z_{i1}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\left(1+\beta x_i\right) \end{array}\right]$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_2}=\sum_{i=1}^{n}\left[\begin{array}{l} \left(z_{i2}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i2}\right)\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right) \\ +\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)\left(z_{i2}\right)\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)\left(z_{i2}\right)\ln\left(1+\beta x_i\right) \end{array}\right].$$

The three estimates $\hat{\beta},\hat{\beta}_1$ and $\hat{\beta}_2$ for parameters $\beta,\beta_1$ and $\beta_2$ can be obtained by solving these three equations.

$$0.95\frac{n}{\beta}+\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln\beta-\frac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_1 z_{i1}}\ln x_i-\frac{\Gamma(0.05)x_i}{1+\beta x_i} \end{bmatrix}=0$$

…(4.2)

$$\sum_{i=1}^{n}\begin{bmatrix} (z_{i1})+\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{1i})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i) \end{bmatrix}=0 \qquad \text{…(4.3)}$$

$$\sum_{i=1}^{n}\begin{bmatrix} (z_{i2})+\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right) \\ +\frac{\beta^{0.95}}{\Gamma(0.05)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)(z_{i2})\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)(z_{i2})\ln(1+\beta x_i) \end{bmatrix}=0$$

…(4.4)

**Case2**: By substitution $\mu_i=\exp(\beta_1 z_{i1}+\beta_3 z_{i3})$ in (4.1)

$$\ln L(x|\mu,\beta)=\sum_{i=1}^{n}\ln\left(\frac{\dfrac{e^{(\beta_1 z_{i1}+\beta_3 z_{i3})}\beta^{0.95}}{\Gamma(0.05)}\beta^{\frac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{0.95}}{\Gamma(0.05)}}x_i^{\left(\frac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}}{\Gamma(0.05)}-1\right)}}{(1+\beta x_i)^{\left[\frac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{0.95}}{\Gamma(0.05)}+1\right]}}\right)$$

Hence, the partial derivatives of the log-likelihood function are

$$\frac{\partial\ln L(x|\mu,\beta)}{\partial\beta}=0.95\frac{n}{\beta}+$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\frac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i-\frac{\Gamma(0.05)x_i}{1+\beta x_i} \end{bmatrix}$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i1}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta_3}=\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i3}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i3}\right)\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\\ +\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln\left(1+\beta x_i\right)\end{array}\right].$$

The three estimates $\hat{\beta},\hat{\beta}_1$ and $\hat{\beta}_3$ for parameters $\beta,\beta_1$ and $\beta_3$ can be obtained by solving these three equations.

$$0.95\frac{n}{\beta}+\frac{1}{\Gamma\left(0.05\right)}\sum_{i=1}^{n}\left[\begin{array}{l}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i}\\ -0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\left(1+\beta x_i\right)\\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i-\dfrac{\Gamma\left(0.05\right)x_i}{1+\beta x_i}\end{array}\right]=0$$

…(4.5)

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i1}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]=0 \qquad \text{…(4.6)}$$

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i3}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i3}\right)\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\\ +\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln\left(1+\beta x_i\right)\end{array}\right]=0. \qquad \text{…(4.7)}$$

**Case3**: By substitution $\mu_i = \exp\left(\beta_1 z_{i1} + \beta_2 z_{i2} + \beta_3 z_{i3}\right)$ in (4.1)

$$\ln L\left(x|\mu,\beta\right) = \sum_{i=1}^{n} \ln \left( \frac{\dfrac{e^{\left(\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}\right)}\beta^{0.95}}{\Gamma(0.05)} \beta^{\frac{e^{\left(\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}\right)}\beta^{0.95}}{\Gamma(0.05)}} x_i^{\left(\frac{e^{\left(\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}\right)}\beta^{0.95}}{\Gamma(0.05)}-1\right)}}{\left(1+\beta x_i\right)^{\left[\frac{e^{\left(\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}\right)}\beta^{0.95}}{\Gamma(0.05)}+1\right]}} \right)$$

Hence, the partial derivatives of the log-likelihood function are

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta} = 0.95\frac{n}{\beta} +$$

$$\frac{1}{\Gamma(0.05)} \sum_{i=1}^{n} \left[ \begin{array}{l} e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \beta^{-0.05} + 0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \beta^{-0.05} \ln\beta - \dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} x_i \beta^{0.95}}{1+\beta x_i} \\ -0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \beta^{-0.05} \ln\left(1+\beta x_i\right) + 0.95\beta^{-0.05} e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln x_i - \dfrac{\Gamma(0.05) x_i}{1+\beta x_i} \end{array} \right]$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta_1} = \sum_{i=1}^{n} \left[ \begin{array}{l} \left(z_{i1}\right) + \dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i1}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i1}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i1}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln\left(1+\beta x_i\right) \end{array} \right]$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta_2} = \sum_{i=1}^{n} \left[ \begin{array}{l} \left(z_{i2}\right) + \dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i2}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln\left(1+\beta x_i\right) \end{array} \right]$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial \beta_3} = \sum_{i=1}^{n} \left[ \begin{array}{l} \left(z_{i3}\right) + \dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i3}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i3}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i3}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} \ln\left(1+\beta x_i\right) \end{array} \right].$$

The four estimates $\hat{\beta}, \hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ for parameters $\beta, \beta_1, \beta_2$ and $\beta_3$ can be obtained by solving these four equations.

$$0.95\frac{n}{\beta}+$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\left[\begin{array}{l} e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln(1+\beta x_i)+0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i-\dfrac{\Gamma(0.05)x_i}{1+\beta x_i} \end{array}\right]=0$$

…(4.8)

$$\sum_{i=1}^{n}\left[\begin{array}{l} (z_{i1})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i) \end{array}\right]=0 \quad \ldots(4.9)$$

$$\sum_{i=1}^{n}\left[\begin{array}{l} (z_{i2})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i) \end{array}\right]=0 \quad \ldots(4.10)$$

$$\sum_{i=1}^{n}\left[\begin{array}{l} (z_{i3})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i) \end{array}\right]=0 \quad \ldots(4.11)$$

Because of the difficulty of solving (4.2)-(4.4), (4.5)-(4.7) and (4.8)-(4.11), with MATLAB, we solve the equations numerically using the Newton-Raphson method to estimate parameters $\beta, \beta_1, \beta_2$ and $\beta_3$. These methods are explained in B.2.2, B.2.3 and B.2.4 of Appendix B.

## 4.3 Results

Table 4.1 shows the estimate of parameters $\beta, \beta_1, \beta_2$ and $\beta_3$ for cases1, 2 and 3.

Table 4.2 shows the SAE, MAE and MSE of Claim Severity for cases1, 2 and 3.

We proved that Case 3, selecting both the age and the gender of the driver, yields the minimum values of SAE, MAE and MSE. For all value of $k$, Case 3 presents the best solution, followed by the results of Case 1 and Case 2. In this study, having tested the $k$ value, we found that the most fitting for GLM employed Pareto model is where $k = 0.80$ because the minimum values of SAE, MAE and MSE. The data consists of the Claim Severity for each policy and several characteristics of the insured person, such as age and gender. Each policy is assumed to have only one claim. Therefore, the expected value of Claim Frequency is equal to 1.

This aim of this research is to solve the problem of insurance pricing of motor insurance claims using the observations from the public non-life insurance companies in Thailand, where the data of Claim Severity are modeled by an Inverse Pareto distribution. We employed GLM and mainly focused on the types of a) age and b) gender, at which are two major rating factors.

Therefore, the age of the driver has more effect on Claim Severity than the gender of the driver.

We can see that increasing the number of risk factors results in decreasing the SAE, MAE and MSE. Therefore, we use the expected value of Claim Severity for both the age and gender to the constitution in calculating the pure premium.

**Table 4.1** The estimate of parameters $\beta, \beta_1, \beta_2$ and $\beta_3$ for cases 1, 2 and 3.

| k | Case Consider | Parameter | | | |
|---|---|---|---|---|---|
| | | $\beta$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| 0.10 | Age | 0.0097 | 11.0009 | -0.0084 | |
| | Gender | 0.0085 | 11.0006 | | -0.0150 |
| | Age and Gender | 0.0110 | 4.6356 | -0.0067 | -0.0087 |
| 0.20 | Age | 0.0097 | 11.0015 | -0.0084 | |
| | Gender | 0.0085 | 11.0010 | | -0.0150 |
| | Age and Gender | 0.0007 | 3.2056 | -0.0048 | -0.0208 |
| 0.30 | Age | 0.0097 | 10.0073 | -0.0084 | |
| | Gender | 0.0085 | 10.0051 | | -0.0148 |
| | Age and Gender | 0.1727 | 7.6537 | -0.0073 | -0.0064 |
| 0.40 | Age | 0.0097 | 10.0133 | -0.0084 | |
| | Gender | 0.0008 | 4.7660 | | 0.0001 |
| | Age and Gender | 0.2179 | 8.1064 | -0.0073 | -0.0063 |
| 0.50 | Age | 0.0077 | 11.0083 | -0.0084 | |
| | Gender | 0.0161 | 11.8047 | | 0.1990 |
| | Age and Gender | 0.0005 | 5.7110 | -0.0047 | -0.0219 |
| 0.60 | Age | 0.0067 | 10.0428 | -0.0083 | |
| | Gender | 0.0078 | 10.0354 | | -0.0137 |
| | Age and Gender | 0.0740 | 8.3812 | -0.0072 | -0.0066 |
| 0.70 | Age | 0.0067 | 10.0938 | -0.0082 | |
| | Gender | 0.0068 | 10.0740 | | -0.0125 |
| | Age and Gender | 0.0638 | 8.8959 | -0.0072 | -0.0067 |
| 0.80 | Age | 0.0070 | 9.4124 | -0.0073 | |
| | Gender | 0.0065 | 9.4532 | | -0.0013 |
| | Age and Gender | 0.0092 | 9.2827 | -0.0070 | -0.0077 |
| 0.90 | Age | 0.0127 | 10.7769 | -0.0079 | |
| | Gender | 0.0058 | 11.2173 | | -0.0088 |
| | Age and Gender | 0.0203 | 10.4978 | -0.0070 | -0.0072 |
| 0.91 | Age | 0.0006 | 10.7742 | -0.0057 | |
| | Gender | 0.0060 | 12.0928 | | -0.0125 |
| | Age and Gender | 0.0765 | 10.7654 | -0.0073 | -0.0065 |
| 0.92 | Age | 0.0146 | 10.9696 | -0.0076 | |
| | Gender | 0.0055 | 10.7429 | | 0.0026 |
| | Age and Gender | 0.0307 | 10.8368 | -0.0071 | -0.0069 |
| 0.93 | Age | 0.0044 | 11.4161 | -0.0075 | |
| | Gender | 0.0002 | 11.3410 | | -0.0075 |
| | Age and Gender | 0.0105 | 14.6688 | -0.1199 | -0.2485 |
| 0.94 | Age | 0.4520 | 11.2278 | -0.0073 | |
| | Gender | 1.4256 | 11.1270 | | 0.0300 |
| | Age and Gender | 0.0917 | 10.7836 | -0.0073 | -0.0064 |

**Table 4.1** The estimate of parameters $\beta, \beta_1, \beta_2$ and $\beta_3$ for cases 1, 2 and 3 (Continued).

| k | Case Consider | Parameter | | | |
|---|---|---|---|---|---|
| | | $\beta$ | $\beta_1$ | $\beta_2$ | $\beta_3$ |
| 0.95 | Age | 0.0205 | 11.3959 | -0.0070 | |
| | Gender | 0.4446 | 11.2589 | | 0.0298 |
| | Age and Gender | 0.0042 | 11.3154 | -0.0063 | -0.0088 |
| 0.96 | Age | $1.30\times10^{2}$ | 12.1005 | -0.0074 | |
| | Gender | 0.0087 | 12.2754 | | -0.0051 |
| | Age and Gender | 0.0498 | 11.7006 | -0.0070 | -0.0070 |
| 0.97 | Age | $6.55\times10^{2}$ | 12.5070 | -0.0074 | |
| | Gender | 0.0113 | 12.3899 | | 0.0008 |
| | Age and Gender | 0.2958 | 12.1000 | -0.0079 | -0.0132 |
| 0.98 | Age | $5.37\times10^{2}$ | 12.6416 | -0.0074 | |
| | Gender | 0.0102 | 12.5794 | | 0.0054 |
| | Age and Gender | 0.1030 | 12.4781 | -0.0070 | -0.0060 |
| 0.99 | Age | 0.0008 | 13.2945 | -0.0048 | |
| | Gender | 0.0096 | 13.0006 | | 0.7010 |
| | Age and Gender | 0.0091 | 13.1738 | -0.0067 | -0.0080 |

**Table 4.2** The SAE, MAE and MSE of Claim Severity for cases 1, 2 and 3.

| K | Cases | SAE | MAE | MSE |
|---|---|---|---|---|
| 0.10 | Age | 47,849,391.49 | 36,920.83 | 2,413,361,350.76 |
| | Gender | 64,829,822.08 | 50,023.01 | 3,405,287,643.18 |
| | Age and Gender | 22,786,388.98 | 17,582.09 | 2,016,096,314.81 |
| 0.20 | Age | 47,877,911.52 | 36,942.83 | 2,414,780,329.70 |
| | Gender | 64,875,081.36 | 50,044.04 | 3,447,286,306.55 |
| | Age and Gender | 22,864,046.68 | 17,654.76 | 2,018,675,914.51 |
| 0.30 | Age | 47,935,002.93 | 36,986.88 | 2,417,625,104.27 |
| | Gender | 64,922,709.67 | 50,094.68 | 3,452,037,612.09 |
| | Age and Gender | 20,190,926.12 | 16,119.54 | 1,964,236,179.39 |
| 0.40 | Age | 21,897,432.33 | 16,896.17 | 1,705,708,087.87 |
| | Gender | 22,738,315.22 | 17,545.00 | 2,014,812,535.18 |
| | Age and Gender | 19,994,345.22 | 15,427.74 | 1,935,256,427.79 |
| 0.50 | Age | 48,202,337.23 | 37,193.16 | 2,431,021,563.78 |
| | Gender | 178,900,886.19 | 138,040.81 | 20,037,427,889.27 |
| | Age and Gender | 22,565,320.02 | 17,411.51 | 2,010,098,139.62 |
| 0.60 | Age | 22,269,672.20 | 17,183.39 | 1,704,670,576.50 |
| | Gender | 26,500,854.43 | 20,448.19 | 1,731,662,136.35 |
| | Age and Gender | 19,376,714.90 | 14,951.17 | 1,911,156,748.23 |
| 0.70 | Age | 49,689,942.21 | 38,341.00 | 2,507,023,474.23 |
| | Gender | 66,716,690.95 | 51,478.93 | 3,584,721,748.93 |
| | Age and Gender | 18,316,858.49 | 14,033.38 | 1,851,289,453.03 |
| 0.80 | Age | 46,863,050.61 | 36,159.76 | 2,366,009,158.11 |
| | Gender | 48,375,509.26 | 37,326.78 | 2,452,070,913.98 |
| | Age and Gender | 18,173,807.27 | 14,023.00 | 1,793,032,546.72 |
| 0.90 | Age | 39,146,772.53 | 30,205.84 | 2,040,966,505.28 |
| | Gender | 81,954,773.38 | 63,236.71 | 4,880,944,140.57 |
| | Age and Gender | 31,246,470.59 | 24,123.82 | 1,810,586,651.30 |
| 0.91 | Age | 42,109,255.76 | 32,491.71 | 2,152,191,562.75 |
| | Gender | 210,340,106.33 | 162,299.46 | 27,158,872,375.02 |
| | Age and Gender | 39,359,761.95 | 30,370.19 | 2,048,229,901.21 |
| 0.92 | Age | 47,718,763.95 | 36,820.03 | 2,405,557,076.69 |
| | Gender | 50,213,699.63 | 38,745.14 | 2,531,419,820.99 |
| | Age and Gender | 42,441,632.32 | 32,748.17 | 2,166,407,468.00 |
| 0.93 | Age | 76,143,692.43 | 58,752.85 | 4,868,270,787.57 |
| | Gender | 93,741,058.31 | 72,331.06 | 6,082,736,406.04 |
| | Age and Gender | 56,049,225.54 | 43,247.88 | 4,614,773,011.84 |

**Table 4.2** The SAE, MAE and MSE of Claim Severity for cases 1, 2 and 3 (Continued).

| K | Cases | SAE | MAE | MSE |
|---|---|---|---|---|
| 0.94 | Age | 62,895,835.58 | 48,530.74 | 3,313,532,325.02 |
| | Gender | 76,361,594.66 | 58,920.98 | 4,371,851,430.74 |
| | Age and Gender | 40,050,534.12 | 30,903.19 | 2,073,483,552.98 |
| 0.95 | Age | 75,961,985.01 | 58,612.64 | 4,349,540,732.77 |
| | Gender | 87,977,841.49 | 67,884.14 | 5,474,576,599.09 |
| | Age and Gender | 71,207,968.47 | 54,944.42 | 3,943,117,041.99 |
| 0.96 | Age | 160,054,575.94 | 123,498.90 | 16,162,478,279.39 |
| | Gender | 256,956,246.85 | 198,268.71 | 40,128,583,220.86 |
| | Age and Gender | 104,996,388.95 | 81,015.73 | 7,428,165,351.74 |
| 0.97 | Age | 247,736,566.97 | 191,154.76 | 37,594,516,256.59 |
| | Gender | 291,270,513.24 | 224,745.77 | 51,355,487,389.70 |
| | Age and Gender | 155,485,746.86 | 119,973.57 | 15,311,728,378.15 |
| 0.98 | Age | 285,814,251.10 | 220,535.69 | 49,796,058,829.97 |
| | Gender | 356,719,113.69 | 275,246.23 | 76,691,793,702.71 |
| | Age and Gender | 242,950,487.33 | 187,461.80 | 36,160,870,453.48 |
| 0.99 | Age | 623,765,849.24 | 481,480.94 | 233,535,757,437.03 |
| | Gender | 915,060,088.95 | 706,064.88 | 547,583,902,818.15 |
| | Age and Gender | 506,582,034.38 | 392,352.66 | 155,818,864,966.15 |

## 4.4 The Application for Prediction of the Insurance Premium

The preceding sections show the construction of GLM. In the following section we are interested in determining the pure premium.

Pure Premium can determine from two components which are frequency and the severity distributions of the potential claims. In our work, we assume each policy has only 1 claim, As a result, the expected value of the Claim Frequencies is equals to 1 $\left(i.e.\ E\left[Y_1\right]=1\right)$. On the other hand, the distribution chosen for modeling severity is the Inverse Pareto distribution. For pricing some insurance, taking the mean of the frequency and the severity distribution produces the pure premium. The model can be written the following form:

$$Frequency = E\left[Y_i\right] = 1,$$

$$Severity = E\left[X_i\right] = \exp\left(\sum_{j}^{p} z_{ij}\beta_j\right),$$

$$Pure\ Premium = E\left[Y_i\right] \times E\left[X_i\right] = \exp\left(\sum_{j}^{p} z_{ij}\beta_j\right).$$

Consequently, the response of the pure premium will be equal to the expected value of the Claim Severity because the expected value of Claim Frequency is equal to 1.

The previous chapter demonstrated the construction of the infinity mixture model that supported the pricing of insurance premium in this Chapter IV. Having achieved the major parts of the Thesis, the Conclusion follows in the next Chapter.

# CHAPTER V

# CONCLUSIONS

This thesis is divided into two parts which are, firstly, the claim modeling for an infinite mixture model and, secondly, the pricing of insurance premiums using GLM which is based on an infinite mixture model for response variables. To verify the concepts, we have used the observations of motor insurance claims for the year 2009. The conclusion, discussion and further research are as follows.

## 5.1 Claim Modeling

### 5.1.1 Conclusion

For the simulations: the group samples are simulated by 99 sample groups of the combination of claim distributions, i.e., Lognormal, Gamma and Weibull distributions. Having stimulated the models, we found that the error is significantly less than that of the classical distribution. For the Application: the classical models, namely, Exponential, Inverse Exponential and Lognormal have been used by actuaries to fit the observations. Having used K-S test for these three classical models, the yield cannot meet the standard of goodness of fit. Finally, we attempted to find a solution by constructing an infinite mixture distribution which becomes superior to that of Inverse Pareto distribution. Thus, the set can be fitted to the modified Inverse Pareto distribution as shown by the K-S test at a significance level of $\alpha = 0.01$.

### 5.1.2 Discussion and Further Research

An infinite mixture model was investigated in this research study which can be fitted to motor insurance claims. The infinite mixture model is useful for some modeling of unobserved heterogeneity in the population and for reducing the problem of the number of components (k) in a finite mixture model.

In further research, a new model can be constructed of infinite mixture distributions which are appropriate to our claim data set. (Please see Appendix E. for the new models) They can be applied to many fields, such as financial data, stock data and for other practical purposes.

## 5.2 Insurance Pricing

### 5.2.1 Conclusion

For the application of the observations, all insurance premiums are based on the GLM which incorporates many risk factors. We found that increasing the number of risk factors resulted in decreasing the SAE, MAE and MSE. Therefore, the expected value of Claim Severity that considers both age and gender is an appropriate model to use for calculating pure premiums. The expected value of Claim Frequency is equal to 1. Therefore, the response of the pure premium will be equal to the expected value of the Claim Severity.

### 5.2.2 Discussion and Further Research

In insurance pricing, the GLM is one methodology which can provide the determination of a pure premium which is dependent on two components, frequency and severity distributions of the potential claims. However, the price of motor insurance policies depends on individual characteristics, such as driver's age, gender, marital status, type of car driven and the age of the vehicle. During the work, we has tried to modify the alternative software packages, i.e., GLIM and R.

Further research resulting from this research study should focus on a generalized linear model with other distributions than the exponential family.

# REFERENCES

# REFERENCES


Arthur, E.R. (1994). Modelling The Claims Process In The Presence Of Covariates. **ASTIN Bulletin**. 24: 265-285.

Brzezniak, Z. and Zastawniak, T. (1999). **Basic Stochastic Processes**. London: Springer-Verlag.

Catherine, F., Merran, E., Nicholas, H. and Brian, P. (2011). **Statistical Distribution**. 4th Edn., John Wiley & Sons Inc.

David, M. (2015). A review of theoretical concepts and empirical literature of non-life insurance pricing. **Procedia Economics and Finance**. 20: 157-162.

Dobson, A.J. (2002). **An Introduction to Generalized Linear Models**. New York: Chapman & Hall/CRC.

Emilio, G.D., Jose, M.S. and Enrique, C.O. (2008). Univariate and multivariate versions of the negative binomail-inverse Gaussian distribution with applications. **Insurance: Mathematics and Economics**. 42: 39-49.

Erisoglu, M., Servi, T., Erisoglu, U. and Calis, N. (2013). Mixture Gamma Distribution for Estimation of Wind Power Potential. **Internationa Journal of Applied Mathematics and Statistics**. 40: 223-231.

Frangos, N. and Karlis, D. (2004). Modelling losses using an exponential-inverse Gaussian distribution. **Insurance: Mathematics and Economics**. 35: 53-67.

Geoff, W. and Serhat, G. (2007). GLM Basic Modelling: Avoiding Common Pitfalls. **Casualty Actuarial Society Forum**. 257-272

Haberman, S. and Renshaw, A.E. (1996). Generalized linear models and actuarial science. **Journal of the  Statisticiain**. 45: 407-436.

Hogg, R.V., Craig, A. and McKean, J.W. (2005). **Introduction to Mathematical Statistic**. 6th Edn., Prentice Hall, Upper Saddle River, NJ.

Jonge, P. and Heller, G.Z. (2008). **Generalized Linear Models for Insurance Data**. New York: Cambridge University Press.

Kart, P.M., Michael, J.B. and Peter, K.W. (2000). Using Generalized Linear Models to Build Dynamic Pricing Systems. **Casualty Actuarial Society Forum**. 107-139.

Klugman, S.A., Panjer, H.H. and Willmot, G.E. (2008). **Loss Models: From Data to Decisions**. 2rd Edn., John Wiley & Sons, Hoboken.

Knight, K. (1999). **Mathematical Statistics**. Texts in Statistical Science Series., Chapman&Hall/CRC.

Mauro, B., Antonello, M. and Lea, P. (2012). **Skew Mixture Models for Loss Distributions: A Bayesian Approach**. MPRA.

McCullagh, P. and Nelder, J.A. (1989). **Generalized Linear Models**. 2nd Edn., Chapman & Hall /CRC.

Mohamed, M.A., Ahmad, M.R. and Noriszura, I. (2010). Approximation of aggregate losses using simulation. **Journal of Mathematics and Statistics**. 6: 233-239.

Ohlsson, E. and Johansson, B. (2010). **Non-life Insurance Pricing with Generalized Linear Models**. Springer-Verlag, Berlin and Heidelberg.

Pacakova, V. and Zapletal, D. (2013). Mixture Modelling of Distributions in Insurance Losses. **Proceedings of the Conference on Applied Mathematics and Computational Methods in Engineeing**. 122-125.

Roosvelt, C. and Mosley, Jr. (2004). Estimating Claim Settlement Values Using GLM. **Casualty Actuarial System**. 291-314.

Sattayatham, P. and Talangtam, T. (2012). Fitting of finite mixture distribution to motor insurance claims. **Journal of Mathematics and Statistics**. 8: 49-56.

Sebah, P. and Gourdon, X. (2001). Newton's method and high order iterations.

Silvie, K. and Lenka, K. (2014). Generalized Linear Models In Vehicle Insurance. **Acta Universitatis Agriculturae et Silviculturae Mendelianae Brunensis**. 62: 383-388.

Stephen, G.K. and Richard, L. (2011). **Risk Models and Their Estimation.** Actex Publications, Inc.

Steven, T.K. (2007). **Numerical Analysis Using MATLAB and Excel.** 3rd Edn., Orchard Publications.

Talangtam, T. (2012). **The Modeling of Loss For Non-life Insurance with Finite Mixture Models of Individual Data**. Ph.D Thesis 2012. Suranaree University of Technology.

Tse, Y.K. (2009). **Nonlife Actuarial Models.** New York: Cambridge University Press.

Werner, G. and Guven, S. (2007). GLM Basic Modelling: Avoiding Common Pitfalls. **Casualty Actuarial Society Forum.** 257-272.

# APPENDICES

# APPENDIX A

# THE CUMULATIVE DISTRIBUTION FUNCTION

This section presents the cdf of the Inverse Pareto distribution $IPa\left(\alpha, \frac{1}{\beta}\right)$.

## The Cumulative Distribution Function

The pdf of Inverse Pareto distribution $IPa\left(\alpha, \frac{1}{\beta}\right)$ is

$$h(x) = \frac{\alpha\beta^{\alpha} x^{\alpha-1}}{(1+\beta x)^{\alpha+1}};\ \alpha, \beta > 0,\ x > 0,$$

Thus,

$$H(x) = \int_0^x f(x) dz = \int_0^x \frac{\alpha\beta^{\alpha} z^{\alpha-1}}{(1+\beta z)^{\alpha+1}} dz$$

Let $t = (1+\beta z)$ then $z = \frac{(t-1)}{\beta}$. If $z = 0 \Rightarrow t = 1$, if $z = x \Rightarrow t = \beta x + 1$

We get $dt = \beta dz$ then $dz = \frac{1}{\beta} dt$.

$$\int_0^x \frac{\alpha\beta^{\alpha} z^{\alpha-1}}{(1+\beta z)^{\alpha+1}} dz = \int_1^{\beta x+1} \frac{\alpha\beta^{\alpha}\left(\frac{t-1}{\beta}\right)^{\alpha-1}}{t^{\alpha+1}} \cdot \frac{1}{\beta} dt$$

$$= \frac{\alpha\beta^{\alpha}}{\beta} \int_{1}^{\beta x+1} \frac{(t-1)^{\alpha-1}}{\beta^{\alpha-1}t^{\alpha+1}}\,dt$$

$$= \frac{\alpha\beta^{\alpha}}{\beta^{\alpha}} \int_{1}^{\beta x+1} \frac{(t-1)^{\alpha-1}}{t^{\alpha-1}\cdot t^{2}}\,dt$$

$$= \alpha \int_{1}^{\beta x+1} \left(\frac{t-1}{t}\right)^{\alpha-1} \frac{1}{t^{2}}\,dt$$

$$= \alpha \int_{1}^{\beta x+1} \left(1-t^{-1}\right)^{\alpha-1} \frac{1}{t^{2}}\,dt$$

$$= \alpha\left[\frac{\left(1-t^{-1}\right)^{\alpha}}{\alpha}\right]_{1}^{\beta x+1} = \left(1-\frac{1}{\beta x+1}\right)^{\alpha}$$

The cdf of Inverse Pareto distribution $IPa\left(\alpha,\frac{1}{\beta}\right)$ is $H(x)=\left(1-\frac{1}{\beta x+1}\right)^{\alpha}$.

# APPENDIX B

# NEWTON RAPHSON METHODS

This section presents the Newton-Raphson Method (See Steven (2007)).

## B.1 Newton Raphson Method

Newton's (or Newton- Raphson) method can be used to approximate the roots of any linear or non-linear equation of any degree. This is an iterative (repetitive procedure) method.

The tangent line (slope) to the curve $y = f(x)$ at the point $(x_1, f(x_1))$.

We assume that the slope is neither zero nor infinite. Then, the slope (first derivative) at $x = x_1$ is

$$f'(x_1) = \frac{y - f(x_1)}{x - x_1}$$

$$y - f(x_1) = f'(x_1)(x - x_1) \qquad \ldots\text{(B.1)}$$

The slope crosses the $x - axis$ at $x = x_2$ and $y = 0.$ Since this point $(x_2, f(x_2)) = (x_2, 0)$ line on the slope line, it satisfies (B.1). By substitution,

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and in general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

## B.2 Newton Raphson Method for Several Variables

(Sebah and Gourdon, 2001) The Newton-Raphson Method is used for estimation of $\theta_1$ and $\theta_2$ for $\hat{\theta}_1$ and $\hat{\theta}_2$ where $\theta_1$ and $\theta_2$ are the parameter of the distribution function.

By The Newton-Raphson Method, starting from the Taylor series of $g_1$ and $g_2$ around the point $(\theta_{10}, \theta_{20})$, the initial values of $\theta_1$ and $\theta_2$ are obtained by the Least Square Method and computed in iteration until they converge to the constants $(\hat{\theta}_1, \hat{\theta}_2)$.

We consider the Taylor series from $g_1$ and $g_2$ around the point $(\theta_{10}, \theta_{20})$.

$$g_1(\theta_1,\theta_2) = g_1(\theta_{10},\theta_{20}) + g_{11}(\theta_{10},\theta_{20})(\theta_1 - \theta_{10}) + g_{12}(\theta_{10},\theta_{20})(\theta_2 - \theta_{20})$$

$$g_2(\theta_1,\theta_2) = g_2(\theta_{10},\theta_{20}) + g_{21}(\theta_{10},\theta_{20})(\theta_1 - \theta_{10}) + g_{22}(\theta_{10},\theta_{20})(\theta_2 - \theta_{20})$$

Such that

$$g_{j1}(\theta_{10},\theta_{20}) = \frac{\partial g_j}{\partial \theta_1}(\theta_1,\theta_2)\Big|_{(\theta_1,\theta_2)=(\theta_{10},\theta_{20})} \;;\; j = 1,2.$$

$$g_{j2}(\theta_{10},\theta_{20}) = \frac{\partial g_j}{\partial \theta_2}(\theta_1,\theta_2)\Big|_{(\theta_1,\theta_2)=(\theta_{10},\theta_{20})} \;;\; j = 1,2.$$

By the Least Square Method, we obtain

$$g_{11}(\theta_1 - \theta_{10}) + g_{12}(\theta_2 - \theta_{20}) = -g_1$$

$$g_{21}(\theta_1 - \theta_{10}) + g_{22}(\theta_2 - \theta_{20}) = -g_2$$

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} \theta_1 - \theta_{10} \\ \theta_2 - \theta_{20} \end{bmatrix} = \begin{bmatrix} -g_1 \\ -g_2 \end{bmatrix}$$

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}\theta_{10}\\ \theta_{20}\end{bmatrix} + \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}^{-1}\begin{bmatrix}-g_1\\ -g_2\end{bmatrix}$$

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}\theta_{10}\\ \theta_{20}\end{bmatrix} + \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\begin{bmatrix}g_{22} & -g_{12}\\ -g_{21} & g_{11}\end{bmatrix}\begin{bmatrix}-g_1\\ -g_2\end{bmatrix}$$

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}\theta_{10}\\ \theta_{20}\end{bmatrix} + \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\begin{bmatrix}g_{12}\cdot g_2 - g_{22}\cdot g_1\\ g_{21}\cdot g_1 - g_{11}\cdot g_2\end{bmatrix}$$

Hence,

$$\theta_1 = \theta_{10} + \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\left(g_{12}\cdot g_2 - g_{22}\cdot g_1\right)$$

$$\theta_2 = \theta_{20} + \frac{1}{g_{11}g_{22}-g_{12}g_{21}}\left(g_{21}\cdot g_1 - g_{11}\cdot g_2\right)$$

By the Newton-Raphson technique, all parameters are simultaneously estimated for each term. The iteration procedure is applied until the values of the parameters do not change or converge to the constants. Finally, we get the estimation value of $(\theta_1, \theta_2)$ to be $(\hat{\theta}_1, \hat{\theta}_2)$ where $\theta_{10}$ and $\theta_{20}$ are the initial parameters to $\theta_1$ and $\theta_2$. Newton-Raphson Method for 3 and 4 variables using the same principle.

In Chapter III, from (3.5)-(3.6), we preferred to solve the equations numerically by using the Newton-Raphson method to estimate parameter $\alpha$ and $\beta$.

### B.2.1 The inverse Pareto distribution

Assume that $X \sim IPa\left(\alpha, \frac{1}{\beta}\right)$ with density

$$h(x) = \frac{\alpha\beta^{\alpha}x^{\alpha-1}}{(1+\beta x)^{\alpha+1}}; \quad \alpha, \beta > 0, \ x > 0.$$

The likelihood function can be written as follows:

$$L(\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha\beta^{\alpha} x_i^{\alpha-1}}{(1+\beta x_i)^{\alpha+1}}; \quad \alpha,\beta > 0,\ x > 0.$$

The log-likelihood function is in the form

$$\ln L(\alpha,\beta) = n\ln\alpha + n\alpha\ln\beta + (\alpha-1)\sum_{i=1}^{n}\ln x_i - (\alpha+1)\sum_{i=1}^{n}\ln(1+\beta x_i)$$

From this the partial derivatives of the log-likelihood function follow:

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\alpha} = \frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+\beta x_i)$$

$$\frac{\partial \ln L(\alpha,\beta)}{\partial\beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i}$$

The two estimations $\hat{\alpha}$ and $\hat{\beta}$ for parameters $\alpha$ and $\beta$ can be obtained by solving these two equations.

$$\frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+\beta x_i) = 0 \qquad \ldots\text{(B.2)}$$

$$\frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} = 0 \qquad \ldots\text{(B.3)}$$

Therefore we solve the B.2 and B.3 by the numerical method using Newton-Raphson, for the estimation parameters $\alpha$ and $\beta$.

By Newton-Raphson; $g_1$, $g_2$, $g_{11}$, $g_{22}$, $g_{12}$

Let $\theta_1 = \alpha$, $\theta_2 = \beta$

$$g_1(\theta_1,\theta_2) = \frac{\partial \ln L(\alpha,\beta)}{\partial\theta_1}$$

$$= \frac{n}{\alpha} + n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln(1+\beta x_i)$$

$$g_{11}(\theta_1,\theta_2) = \frac{\partial^2 \ln L(\alpha,\beta)}{\partial^2\theta_1}$$

$$= \frac{\partial}{\partial\alpha}\left(\frac{n}{\alpha}+n\ln\beta+\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\ln(1+\beta x_i)\right) = \frac{-n}{\alpha^2}$$

$$g_2(\theta_1,\theta_2) = \frac{\partial\ln L(\alpha,\beta)}{\partial\theta_2}$$

$$= \frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)$$

$$g_{22}(\theta_1,\theta_2) = \frac{\partial^2\ln L(\alpha,\beta)}{\partial^2\theta_2}$$

$$= \frac{\partial}{\partial\beta}\left(\frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)\right)$$

$$= \frac{-n\alpha}{\beta^2}+(\alpha+1)\sum_{i=1}^{n}\frac{x_i^2}{(1+\beta x_i)^2}$$

$$g_{21}(\theta_1,\theta_2) = g_{12}(\theta_1,\theta_2) = \frac{\partial^2\ln L(\alpha,\beta)}{\partial\theta_1\partial\theta_2}$$

$$= \frac{\partial}{\partial\alpha}\left(\frac{n\alpha}{\beta}-(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i}{1+\beta x_i}\right)\right)$$

```
%-------------------------------------------------------
%--------------------New distribution-------------------
%-------------------------------------------------------
clc;

s=load('D:DATA\Programming\OLAY_Program\Data5.txt');

ss=zeros; z=zeros; ms=zeros;
Lss=zeros;

n=1296;

for i=1:n
    ss(i)=s(i,1);

end

z=sort(ss);

%--------------------------------------------------------
%-------------- MLE of InverseExponential ---------------
%--------------------------------------------------------

a_ss=zeros;

for i=1:n
    a_ss(i)=1/ss(i);
end

Lss=a_ss;
M=n/sum(Lss)

%--------------------------------------------------------

FF=zeros;

for i=1:n
    FF(i)=exp(-M/z(i));
end

%------------ K-S test of InverseExponential --------------

FM=zeros;

for k=1:n
    FM(k)=k/n;
end

DIF=FM-FF
D=Max(DIF)
```

```
%------------------------------------------------------
%---------- Newton Raphson Of New Distribution ----------
%------------------------------------------------------

Par1=-100;
Par2=-100;

%==================== Set initial value =================

v=0.000011;
M=0.52286;

%----------------- g1,g11,g12,g21,g2,g22 -----------------

niter = 0;

while ((abs(M-Par1)>0.00001) || (abs(v-Par2)>0.00001))

DET=0;

   g1=0;g11=0;g12=0;g21=0;g2=0;g22=0;
   Xg1=0;Xg2=0;Xg11=0;Xg12=0;Xg21=0;Xg22=0;

   if niter==0
       M=M;
       v=v;
   else
        M=Par1;
        v=Par2;
   end

GG1=0;GG1_1=0;b_b=zeros;c_c=zeros;

for i=1:n
    b_b(i)=log(z(i));
    c_c(i)=log(1+v*z(i));
end

GG1=sum(b_b);
GG1_1=sum(c_c);

Xg1= n/M+n*log(v)+GG1-GG1_1;
Xg11= -n/(M^2);

GG2=0;d_d=zeros;

for i=1:n
      d_d(i)=z(i)/(1+v*z(i));
end

GG2=sum(d_d);
Xg2= (n*M)/v-(M+1)*GG2;
GG22= 0; e_e=zeros;

for i=1:n
      e_e(i)= (z(i)^2)/(1+v*z(i))^2;
```

```
end

GG22=sum(e_e);
Xg22= (-n*M)/(v^2)+(M+1)*GG22;
GG21= 0; f_f=zeros;

for i=1:n
      f_f(i)=z(i)/(1+v*z(i));
end

GG21=sum(f_f);
Xg21= n/v-GG21;
Xg12=Xg21;

%--------------------------------------------------------

g1=Xg1 ;
g11=Xg11;
g12=Xg12;
g21=Xg21;
g2=Xg2;
g22=Xg22;

%===================== Check for DET =====================

DET=(g11*g22)-(g12*g21);

%--------------------------------------------------------

Par1= M + ((g12*g2)-(g22*g1))/DET
Par2= v +((g1*g21)-(g2*g11))/DET

     niter = niter + 1

     diff_Par1=abs(M-Par1)
     diff_Par2=abs(v-Par2)
end

iteration = niter

disp('===== Par1 & Par2 =====');

digits(10)
disp(vpa(Par1));
disp(vpa(Par2));


%------------------ End of Newton Raphson -------------------
```

```
%------------------- For new distribution -------------------

FF_new=zeros;

Par1
Par2

for i=1:n
   a1=Par2*z(i)+1;
   b1=1/a1;
   c1=1-b1;
   d1=c1^Par1;
   FF_new(i)=d1;
end

FF_new;


%---------------K-S test of new distribution--------------

FM=zeros;

for k=1:n
   FM(k)=k/n;
End

DIF=FM-FF_new;

D_new=Max(DIF)

D % to comparison
```

In Chapter IV, from (4.2)-(4.4), (4.5)-(4.7) and (4.8)-(4.11), we preferred to solve the equations numerically by using the Newton-Raphson Method to estimate parameters $\beta, \beta_1, \beta_2$ and $\beta_3$.

**B.2.2 Newton Raphson Method for estimation of $\beta, \beta_1$ and $\beta_2$.**

The three estimation $\hat{\beta}, \hat{\beta}_1$ and $\hat{\beta}_2$ for parameters $\beta, \beta_1$ and $\beta_2$ can be obtained by solving these three equations.

$$0.95\frac{n}{\beta}+\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_1 z_{i1}}\ln x_i-\dfrac{\Gamma(0.05)x_i}{1+\beta x_i} \end{bmatrix}=0$$

…(B.4)

$$\sum_{i=1}^{n}\begin{bmatrix} (z_{i1})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{1i})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i) \end{bmatrix}=0$$

…(B.5)

$$\sum_{i=1}^{n}\begin{bmatrix} (z_{i2})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)+\dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)(z_{i2})\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}\left(e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\right)(z_{i2})\ln(1+\beta x_i) \end{bmatrix}=0$$

…(B.6)

Therefore we solve the (B.4) - (B.6) by the numerical method using Newton-Raphson, for the estimation parameters $\beta, \beta_1$ and $\beta_2$.

By Newton-Raphson; $g_1$, $g_2$, $g_3$, $g_{11}$, $g_{12}=g_{21}$, $g_{13}=g_{31}$, $g_{22}$, $g_{23}=g_{32}$, $g_{33}$.

Let $\theta_1 = \beta,\ \theta_2 = \beta_1,\ \theta_3 = \beta_2$

$$g_1(\theta_1,\theta_2,\theta_3) = \frac{\partial \ln L(x|\mu,\beta)}{\partial \theta_1} = \frac{\partial \ln L(x|\mu,\beta)}{\partial \beta}$$

$$\frac{\partial \ln L(x|\mu,\beta)}{\partial \beta}$$

$$= 0.95\frac{n}{\beta} + \frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05} + 0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln\beta - \dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}} x_i \beta^{0.95}}{1+\beta x_i} \\ -0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\beta^{-0.05}\ln(1+\beta x_i) + 0.95\beta^{-0.05} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i - \dfrac{\Gamma(0.05) x_i}{1+\beta x_i} \end{bmatrix}$$

$$g_{11}(\theta_1,\theta_2,\theta_3) = \frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \theta_1 \partial \theta_1} = \frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \beta \partial \beta}$$

$$\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \beta \partial \beta} = -0.95 n \beta^{-2}$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} \left(-0.05\beta^{-1.05}\right) e^{\beta_1 z_{i1}+\beta_2 z_{i2}} + 0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\left(\beta^{-1.05} - 0.05\beta^{-1.05}\ln\beta\right) \\ -e^{\beta_1 z_{i1}+\beta_2 z_{i2}} x_i \left(\dfrac{0.95\beta^{-0.05} - 0.05\beta^{0.95}(x_i)}{(1+\beta x_i)^2}\right) \\ -0.95 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\left(\dfrac{\beta^{-0.05} x_i}{1+\beta x_i} - 0.05\beta^{-1.05}\ln(1+\beta x_i)\right) \\ -0.0475\beta^{-1.05} e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i + \dfrac{\Gamma(0.05)(x_i)^2}{(1+\beta x_i)^2} \end{bmatrix}$$

$$g_{12}(\theta_1,\theta_2,\theta_3) = \frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \theta_2 \partial \theta_1} = \frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \beta_1 \partial \beta} = g_{21}(\theta_1,\theta_2,\theta_3)$$

$$\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial \beta_1 \partial \beta} = \frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} \beta^{-0.05}(z_{i1}) e^{\beta_1 z_{i1}+\beta_2 z_{i2}} + 0.95\beta^{-0.05}(z_{1i}) e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\beta \\ -\dfrac{\beta^{0.95} x_i}{1+\beta x_i}(z_{1i}) e^{\beta_1 z_{i1}+\beta_2 z_{i2}} - 0.95\beta^{-0.05}(z_{1i}) e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}(z_{1i}) e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i \end{bmatrix}$$

$$g_{13}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3\partial\theta_1}=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_2\partial\beta}=g_{31}\left(\theta_1,\theta_2,\theta_3\right)$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_2\partial\beta}=\frac{1}{\Gamma\left(0.05\right)}\sum_{i=1}^{n}\left[\begin{array}{l}\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\beta\\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}-0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\left(1+\beta x_i\right)\\ +0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\end{array}\right]$$

$$g_2\left(\theta_1,\theta_2,\theta_3,\theta_4\right)=\frac{\partial l}{\partial\theta_2}=\frac{\partial l}{\partial\beta_1}$$

$$\frac{\partial l}{\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{1i}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_{22}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 l}{\partial\theta_2\partial\theta_2}=\frac{\partial^2 l}{\partial\beta_1\partial\beta_1}$$

$$\frac{\partial^2 l}{\partial\beta_1\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_{23}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 l}{\partial\theta_3\partial\theta_2}=\frac{\partial^2 l}{\partial\beta_2\partial\beta_1}=g_{32}\left(\theta_1,\theta_2,\theta_3\right)$$

$$\frac{\partial^2 l}{\partial\beta_2\partial\beta_1}=\sum_{i=1}^{n}\begin{bmatrix}\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i1})(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i)\end{bmatrix}$$

$$g_3(\theta_1,\theta_2,\theta_3)=\frac{\partial l}{\partial\theta_3}=\frac{\partial l}{\partial\beta_2}$$

$$\frac{\partial l}{\partial\beta_2}=\sum_{i=1}^{n}\begin{bmatrix}(z_{i2})+\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i)\end{bmatrix}$$

$$g_{33}(\theta_1,\theta_2,\theta_3)=\frac{\partial^2 l}{\partial\theta_3\partial\theta_3}=\frac{\partial^2 l}{\partial^2\beta_2}$$

$$\frac{\partial^2 l}{\partial^2\beta_2}=\sum_{i=1}^{n}\begin{bmatrix}\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln x_i\\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}}\ln(1+\beta x_i)\end{bmatrix}$$

### B.2.3 Newton Raphson Method for estimation of $\beta,\beta_1$ and $\beta_3$.

The three estimations $\hat{\beta},\hat{\beta}_1$ and $\hat{\beta}_2$ for parameters $\beta,\beta_1$ and $\beta_3$ can be obtained by solving these three equations.

$$0.95\frac{n}{\beta}+\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\frac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i}\\ -0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln(1+\beta x_i)\\ +0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i-\frac{\Gamma(0.05)x_i}{1+\beta x_i}\end{bmatrix}=0$$

…(B.7)

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i1}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]=0$$

…(B.8)

$$\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i3}\right)+\dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i3}\right)\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)+\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln x_i\\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\right)\left(z_{i3}\right)\ln\left(1+\beta x_i\right)\end{array}\right]=0$$

…(B.9)

Therefore we solve the (B.7) - (B.9) by the numerical method using Newton-Raphson, for the estimation parameters $\beta, \beta_1$ and $\beta_3$.

By Newton-Raphson; $g_1,\ g_2,\ g_3,\ g_{11},\ g_{12}=g_{21},\ g_{13}=g_{31},\ g_{22},\ g_{23}=g_{32},\ g_{33}$.

Let $\theta_1=\beta,\ \theta_2=\beta_1,\ \theta_3=\beta_3$

$$g_1\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial l}{\partial\theta_1}=\frac{\partial l}{\partial\beta}$$

$$\frac{\partial\ln L\left(x|\mu,\beta\right)}{\partial\beta}$$

$$=0.95\frac{n}{\beta}+\frac{1}{\Gamma\left(0.05\right)}\sum_{i=1}^{n}\left[\begin{array}{l}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i}\\ -0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\beta^{-0.05}\ln\left(1+\beta x_i\right)+0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i-\dfrac{\Gamma\left(0.05\right)x_i}{1+\beta x_i}\end{array}\right]$$

$$g_{11}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 l}{\partial\theta_1\partial\theta_1}=\frac{\partial^2 l}{\partial\beta\partial\beta}$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta\partial\beta}$$

$$= -0.95n\beta^{-2} + \frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} \left(-0.05\beta^{-1.05}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}} + 0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\left(\beta^{-1.05} - 0.05\beta^{-1.05}\ln\beta\right) \\ -e^{\beta_1 z_{i1}+\beta_3 z_{i3}}x_i\left(\dfrac{0.95\beta^{-0.05} - 0.05\beta^{0.95}(x_i)}{(1+\beta x_i)^2}\right) \\ -0.95e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\left(\dfrac{\beta^{-0.05}x_i}{1+\beta x_i} - 0.05\beta^{-1.05}\ln(1+\beta x_i)\right) \\ -0.0475\beta^{-1.05}e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i + \dfrac{\Gamma(0.05)(x_i)^2}{(1+\beta x_i)^2} \end{bmatrix}$$

$$g_{12}(\theta_1,\theta_2,\theta_3) = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_2\partial\theta_1} = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta} = g_{21}(\theta_1,\theta_2,\theta_3)$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta} = \frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} \beta^{-0.05}(z_{i1})e^{\beta_1 z_{i1}+\beta_3 z_{i3}} + 0.95\beta^{-0.05}(z_{i1})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\beta \\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}(z_{i1})e^{\beta_1 z_{i1}+\beta_3 z_{i3}} - 0.95\beta^{-0.05}(z_{i1})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}(z_{i1})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i \end{bmatrix}$$

$$g_{13}(\theta_1,\theta_2,\theta_3) = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3\partial\theta_1} = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3\partial\beta} = g_{31}(\theta_1,\theta_2,\theta_3)$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3\partial\beta} = \frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix} \beta^{-0.05}(z_{i3})e^{\beta_1 z_{i1}+\beta_3 z_{i3}} + 0.95\beta^{-0.05}(z_{i3})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\beta \\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}(z_{i3})e^{\beta_1 z_{i1}+\beta_3 z_{i3}} - 0.95\beta^{-0.05}(z_{i3})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln(1+\beta x_i) \\ +0.95\beta^{-0.05}(z_{i3})e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i \end{bmatrix}$$

$$g_2(\theta_1,\theta_2,\theta_3) = \frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_2} = \frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1}$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i1}\right)+\frac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\-\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_{22}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_2\partial\theta_2}=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta_1}$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\-\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_{23}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3\partial\theta_2}=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3\partial\beta_1}=g_{32}\left(\theta_1,\theta_2,\theta_3\right)$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3\partial\beta_1}=\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\-\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_3\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3}=\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3}$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3}=\sum_{i=1}^{n}\left[\begin{array}{l}\left(z_{i3}\right)+\frac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\-\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

$$g_{33}\left(\theta_1,\theta_2,\theta_3\right)=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3\partial\theta_3}=\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial^2\beta_3}$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial^2\beta_3}=\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i3}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i3}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln x_i\\-\frac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i3}\right)^2 e^{\beta_1 z_{i1}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right)\end{array}\right]$$

**B.2.4 Newton Raphson Method for estimation of $\beta, \beta_1, \beta_2$ and $\beta_3$.**

The four estimations $\hat{\beta}, \hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ for parameters $\beta, \beta_1, \beta_2$ and $\beta_3$. can be obtained by solving these four equations.

$$0.95\frac{n}{\beta}+$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\left[\begin{array}{l} e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln(1+\beta x_i)+0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i-\dfrac{\Gamma(0.05)x_i}{1+\beta x_i}\end{array}\right]=0$$

…(B.10)

$$\sum_{i=1}^{n}\left[\begin{array}{l}(z_{i1})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i1})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i)\end{array}\right]=0$$

…(B.11)

$$\sum_{i=1}^{n}\left[\begin{array}{l}(z_{i2})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i2})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i)\end{array}\right]=0$$

…(B.12)

$$\sum_{i=1}^{n}\left[\begin{array}{l}(z_{i3})+\dfrac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i)\end{array}\right]=0$$

…(B.13)

Therefore we solve the (B.10) – (B.13) by the numerical method using Newton-Raphson, for the estimation of the paramaters $\beta, \beta_1, \beta_2$ and $\beta_3$.

By Newton Raphson; $g_1,\ g_2,\ g_3,\ g_4,\ g_{11},\ g_{12}=g_{21},\ g_{13}=g_{31},\ g_{14}=g_{41},\ g_{22},\ g_{23}=g_{32}$

$g_{24}=g_{42}, g_{33},\ g_{34}=g_{43},\ g_{44}.$

Let $\theta_1=\beta,\ \theta_2=\beta_1,\ \theta_3=\beta_2,\ \theta_4=\beta_3$

$$g_1(\theta_1,\theta_2,\theta_3,\theta_4)=\frac{\partial l}{\partial\theta_1}=\frac{\partial l}{\partial\beta}$$

$$\frac{\partial \ln L(x|\mu,\beta)}{\partial\beta}=0.95\frac{n}{\beta}+$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\left[\begin{array}{l} e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln\beta-\dfrac{e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}x_i\beta^{0.95}}{1+\beta x_i} \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\beta^{-0.05}\ln(1+\beta x_i)+0.95\beta^{-0.05}e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i-\dfrac{\Gamma(0.05)x_i}{1+\beta x_i}\end{array}\right]$$

$$g_{11}(\theta_1,\theta_2,\theta_3,\theta_4)=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\theta_1\partial\theta_1}=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\beta\partial\beta}$$

$$\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\beta\partial\beta}=-0.95n\beta^{-2}$$

$$\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\left[\begin{array}{l} \left(-0.05\beta^{-1.05}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\left(\beta^{-1.05}-0.05\beta^{-1.05}\ln\beta\right) \\ -e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}x_i\left(\dfrac{0.95\beta^{-0.05}-0.05\beta^{0.95}(x_i)}{(1+\beta x_i)^2}\right) \\ -0.95e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\left(\dfrac{\beta^{-0.05}x_i}{1+\beta x_i}-0.05\beta^{-1.05}\ln(1+\beta x_i)\right) \\ -0.0475\beta^{-1.05}e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i+\dfrac{\Gamma(0.05)(x_i)^2}{(1+\beta x_i)^2}\end{array}\right]$$

$$g_{12}(\theta_1,\theta_2,\theta_3,\theta_4)=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\theta_2\partial\theta_1}=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\beta_1\partial\beta}=g_{21}(\theta_1,\theta_2,\theta_3,\theta_4)$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_1\partial\beta}$$

$$=\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix}\beta^{-0.05}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+0.95\beta^{-0.05}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\beta \\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}-0.95\beta^{-0.05}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \\ +0.95\beta^{-0.05}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i\end{bmatrix}$$

$$g_{13}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)=\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\theta_3\partial\theta_1}=\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_2\partial\beta}=g_{31}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_2\partial\beta}$$

$$=\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix}\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\beta \\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}-0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \\ +0.95\beta^{-0.05}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i\end{bmatrix}$$

$$g_{14}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)=\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\theta_4\partial\theta_1}=\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_3\partial\beta}=g_{41}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_3\partial\beta}$$

$$=\frac{1}{\Gamma(0.05)}\sum_{i=1}^{n}\begin{bmatrix}\beta^{-0.05}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+0.95\beta^{-0.05}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\beta \\ -\dfrac{\beta^{0.95}x_i}{1+\beta x_i}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}-0.95\beta^{-0.05}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \\ +0.95\beta^{-0.05}\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i\end{bmatrix}$$

$$g_2\left(\theta_1,\theta_2,\theta_3,\theta_4\right)=\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial\theta_2}=\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial\beta_1}$$

$$\frac{\partial \ln L\left(x\middle|\mu,\beta\right)}{\partial \beta_1} = \sum_{i=1}^{n} \begin{bmatrix} \left(z_{i1}\right) + \dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{bmatrix}$$

$$g_{22}\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_2\partial\theta_2} = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta_1}$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_1\partial\beta_1} = \sum_{i=1}^{n} \begin{bmatrix} \dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{bmatrix}$$

$$g_{23}\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_3\partial\theta_2} = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_2\partial\beta_1} = g_{32}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)$$

$$\frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_2\partial\beta_1}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} \dfrac{\beta^{0.95}\ln\beta}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\dfrac{\beta^{0.95}}{\Gamma\left(0.05\right)}\left(z_{i1}\right)\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{bmatrix}$$

$$g_{24}\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\theta_4\partial\theta_2} = \frac{\partial^2 \ln L\left(x\middle|\mu,\beta\right)}{\partial\beta_3\partial\beta_1} = g_{42}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_3\partial\beta_1} = \sum_{i=1}^{n}\left[\begin{array}{l} \frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i1}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{array}\right]$$

$$g_3\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial\theta_3} = \frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial\beta_2}$$

$$\frac{\partial \ln L\left(x|\mu,\beta\right)}{\partial\beta_2} = \sum_{i=1}^{n}\left[\begin{array}{l} \left(z_{i2}\right)+\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{array}\right]$$

$$g_{33}\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\theta_3\partial\theta_3} = \frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial^2\beta_2}$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial^2\beta_2} = \sum_{i=1}^{n}\left[\begin{array}{l} \frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i2}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{array}\right]$$

$$g_{34}\left(\theta_1,\theta_2,\theta_3,\theta_4\right) = \frac{\partial^2 \ln L\left(x|\mu,\chi\right)}{\partial\theta_4\partial\theta_3} = \frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_3\partial\beta_2} = g_{43}\left(\theta_1,\theta_2,\theta_3,\theta_4\right)$$

$$\frac{\partial^2 \ln L\left(x|\mu,\beta\right)}{\partial\beta_3\partial\beta_2} = \sum_{i=1}^{n}\left[\begin{array}{l} \frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}\left(z_{i2}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}} + \frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i \\ -\frac{\beta^{0.95}}{\Gamma(0.05)}\left(z_{i2}\right)\left(z_{i3}\right)e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln\left(1+\beta x_i\right) \end{array}\right]$$

$$g_4(\theta_1,\theta_2,\theta_3,\theta_4)=\frac{\partial \ln L(x|\mu,\beta)}{\partial\theta_4}=\frac{\partial \ln L(x|\mu,\beta)}{\partial\beta_3}$$

$$\frac{\partial \ln L(x|\mu,\beta)}{\partial\beta_3}=\sum_{i=1}^{n}\left[\begin{array}{l}(z_{i3})+\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i\\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i)\end{array}\right]$$

$$g_{44}(\theta_1,\theta_2,\theta_3,\theta_4)=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\theta_4\partial\theta_4}=\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial\beta_3\partial\beta_3}$$

$$\frac{\partial^2 \ln L(x|\mu,\beta)}{\partial^2\beta_3}=\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\beta^{0.95}\ln\beta}{\Gamma(0.05)}(z_{i3})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}+\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln x_i\\ -\frac{\beta^{0.95}}{\Gamma(0.05)}(z_{i3})^2 e^{\beta_1 z_{i1}+\beta_2 z_{i2}+\beta_3 z_{i3}}\ln(1+\beta x_i)\end{array}\right]$$

# APPENDIX C

# DISTRIBUTION

This section presents some special probability distributions, including definitions, criteria and material for our simulation and model fitting. We briefed from some references, that are as follows:

1) Klugman, S.A., Panjer, H.H. and Willmot, G.E. (2008). Loss Models: From Data to Decisions.
2) http://www.math.uah.edu.
3) http://en.wikipedia.org/wiki/P-P plot.
4) http://wiki.math.yorku.ca.

## C.1 Loss Distributions

### C.1.1 Lognormal distribution

A random variable $X$ is said to be Lognormal distributed with parameter $\mu$ and $\sigma$ denoted by $X \sim LN(\mu, \sigma)$.

CDF : $F_X(x) = \Phi\left(\dfrac{\ln x - \mu}{\sigma}\right); \quad \mu \in R,\ \sigma > 0,\ x > 0.$

PDF : $f_X(x) = \dfrac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\dfrac{(\ln x - \mu)^2}{2\sigma^2}\right),$

Moment: $E\left[X^k\right] = \exp\left(k\mu + \dfrac{1}{2}k^2\sigma^2\right).$

### C.1.2 Gamma Distribution

A random variable $X$ is said to be Gamma distributed with parameter $\theta$ denoted by $X \sim Gamma(\alpha, \theta)$.

CDF : $F_X(x) = \Gamma\left(\alpha; \dfrac{x}{\theta}\right); \quad \alpha, \theta > 0, \; x > 0.$

PDF : $f_X(x) = \dfrac{x^{\alpha-1}}{\theta^{\alpha}\Gamma(\alpha)} \exp\left(-\dfrac{x}{\theta}\right)$

Moment: $E\left[X^k\right] = \dfrac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k > -\alpha.$

### C.1.3 Weibull Distribution

A random variable $X$ is said to be Inverse exponentailly distributed with parameter $\theta$ and $\tau$ denoted by $X \sim Wei(\theta, \tau)$.

CDF : $F_X(x) = 1 - \exp\left[-\left(x/\theta\right)^{\tau}\right]; \quad \theta, \tau > 0, \; x > 0.$

PDF : $f_X(x) = \dfrac{\tau\left(x/\theta\right)^{\tau}}{x} \exp\left(-\dfrac{x}{\theta}\right)^{\tau}$

Moment: $E\left[X^k\right] = \theta^k \Gamma\left(\alpha + \dfrac{k}{\tau}\right), \quad k > -\alpha.$

## C.2 Skewness Newton

Suppose that $X$ is a real-valued random variable for the experiment. We will let $\mu = E[X]$ and $\sigma^{2} = \mathrm{var}(X)$.

The skewness of $X$ is the third moment of the standard score of $X$ :

$$skew(X) = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

The distribution of $X$ is said to be positively skewed when the probability density function has a long tail to the right, if the distribution is negatively skewed then the probability density function has a long tail to the left. A symmetric distribution is unskewed. (A normal distribution has a skewness equal to $0.$)

## C.3 The Simulation

The simulated data by combinations of claim distributions. Each component mixed has the same number of claims. The simulated data by composed parameters of claim distributions as shown on Table C.1 and Table C.2.

**Table C.1** The 2 mixed components.

| Parameters | Distributions |
|---|---|
| Lognormal/Lognormal<br>$(\mu=5,\ \sigma=2), (\mu=6,\ \sigma=1)$ | Lognormal/Gamma<br>$(\mu=5,\ \sigma=2), (\alpha=60000,\ \beta=3)$ |
| Gamma/Gamma<br>$(\alpha=2500,\ \beta=1), (\alpha=50000,\ \beta=3)$ | Lognormal/Weibull<br>$(\mu=6,\ \sigma=1), (c=2500,\ \tau=1)$ |
| Weibull/Weibull<br>$(c=2000,\ \tau=2),\ (c=60000,\ \tau=3)$ | |

**Table C.2** The 3 mixed components.

| Parameter | Distributions |
|---|---|
| Lognormal/ Lognormal/ Lognormal $(\mu=6,\ \sigma=1),(\mu=8,\ \sigma=2),$ $(\mu=10,\ \sigma=3)$ | Lognormal/ Gamma/ Weibull $(\mu=8,\ \sigma=2),\ (\alpha=2000,\ \beta=1),$ $(c=80000,\ \tau=4)$ |
| Gamma /Gamma/ Gamma $(\alpha=2000,\ \beta=1),\ (\alpha=40000,\ \beta=2),$ $(\alpha=80000,\ \beta=1)$ | |
| Weibull/ Weibull/ Weibull $(c=2000,\ \tau=2),\ (c=60000,\ \tau=3),$ $(c=80000,\ \tau=4)$ | |

## C.4 Levels of Singnificance for the K-S Test.

Table C.3 below lists the singnificance level ($\alpha$) for a test statistic $D$ as employed in the K-S test.

**Table C.3** The level of significance for *D*.

| Sample size ($n$) | Level of significance ($\alpha$) for $D$ | | | | |
|---|---|---|---|---|---|
| | 0.2 | 0.15 | 0.1 | 0.05 | 0.01 |
| 1 | 0.900 | 0.925 | 0.950 | 0.975 | 0.995 |
| 2 | 0.684 | 0.726 | 0.776 | 0.842 | 0.929 |
| 3 | 0.565 | 0.597 | 0.642 | 0.708 | 0.828 |
| 4 | 0.494 | 0.525 | 0.564 | 0.624 | 0.733 |
| 5 | 0.446 | 0.474 | 0.510 | 0.565 | 0.669 |
| 6 | 0.410 | 0.436 | 0.470 | 0.521 | 0.618 |
| 7 | 0.381 | 0.405 | 0.438 | 0.486 | 0.577 |
| 8 | 0.358 | 0.381 | 0.411 | 0.457 | 0.543 |
| 9 | 0.339 | 0.360 | 0.388 | 0.432 | 0.514 |
| 10 | 0.322 | 0.342 | 0.368 | 0.410 | 0.490 |
| 11 | 0.307 | 0.326 | 0.352 | 0.391 | 0.468 |
| 12 | 0.295 | 0.313 | 0.338 | 0.375 | 0.450 |
| 13 | 0.284 | 0.302 | 0.325 | 0.361 | 0.433 |
| 14 | 0.274 | 0.292 | 0.314 | 0.349 | 0.418 |
| 15 | 0.266 | 0.283 | 0.304 | 0.338 | 0.404 |
| 16 | 0.258 | 0.274 | 0.295 | 0.328 | 0.392 |
| 17 | 0.250 | 0.266 | 0.286 | 0.318 | 0.381 |
| 18 | 0.244 | 0.259 | 0.278 | 0.309 | 0.371 |
| 19 | 0.237 | 0.252 | 0.272 | 0.301 | 0.363 |
| 20 | 0.231 | 0.246 | 0.264 | 0.294 | 0.356 |
| 25 | 0.210 | 0.220 | 0.240 | 0.270 | 0.320 |
| 30 | 0.190 | 0.200 | 0.220 | 0.240 | 0.290 |
| 35 | 0.180 | 0.190 | 0.210 | 0.230 | 0.270 |
| Over 35 | $\frac{1.07}{\sqrt{n}}$ | $\frac{1.14}{\sqrt{n}}$ | $\frac{1.22}{\sqrt{n}}$ | $\frac{1.36}{\sqrt{n}}$ | $\frac{1.63}{\sqrt{n}}$ |

## C.5 P-P plot.

In statistics, a P-P plot (probability-probability plot or precent-precent plot) is used to see if a given set of data follows some specified distribution. It should be approximately linear if the specified distribution is the correct model.

A P-P plot compares the theoretical cumulative distribution function, $F(\cdot)$, of the specified model with the empirical cumulative distribution function (ECDF) of data. The ECDF, $F_n(x)$, is defined as the proportion of non-missing observations less than or equal to $x$, so that $F_n(x_i) = \frac{i}{n}$.

# APPENDIX D

# THE TEST OF DATA

In Chapter IV, we will use Program R to test for Normality and Nonlinearity.

**Program R**

```
> rm(list=ls())
> library(tseries)
> data <- read.table("Claim_data.txt", header = TRUE)
> y <- as.ts(data$Y)
> #Test for Normality
> shapiro.test(y)
	Shapiro-Wilk normality test
data:  y
W = 0.95441, p-value < 2.2e-16

> #Computes the Augmented Dickey-Fuller test for the null that x has a unit root.
> adf.test(y)
	Augmented Dickey-Fuller Test
data:  y
Dickey-Fuller = -11.519, Lag order = 10, p-value = 0.01
alternative hypothesis: stationary
Warning message:
In adf.test(y) : p-value smaller than printed p-value

> #ComputestheKwiatkowski-Phillips-Schmidt-Shin (KPSS) test for the null
hypothesis that x is level or trend stationary.
> kpss.test(y)
	KPSS Test for Level Stationarity
data:  y
KPSS Level = 0.14183, Truncation lag parameter = 8, p-value = 0.1
Warning message:
In kpss.test(y) : p-value greater than printed p-value
```

```
> #Computes the Phillips-Perron test for the null hypothesis that x has a unit root.
> pp.test(y)
        Phillips-Perron Unit Root Test
data:  y
Dickey-Fuller Z (alpha) = -1283.4, Truncation lag parameter = 7, p-value= 0.01
alternative hypothesis: stationary
Warning message:
In pp.test(y) : p-value smaller than printed p-value
```

# APPENDIX E

# NEW DISTRIBUTION

In this section, we present some new models for the claim modeling. An infinite mixture distribution is the methods used to obtain new distributions.

## E.1 Loss Distributions

**Lognormal distribution**

A random variable $X$ is said to be Lognormally distributed with parameter $\mu$ and $\sigma$ denoted by $X \sim LN(\mu, \sigma)$.

$$\text{PDF} \quad : \quad f_X(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right); \quad \mu \in R, \sigma > 0, x > 0.$$

**Exponential distribution**

A random variable $\Theta$ is said to be Exponentially distributed with parameter $\theta$ denoted by $\Theta \sim Exp(\theta)$.

$$\text{PDF} \quad : \quad g(\mu) = \frac{1}{\theta} \exp\left(-\frac{\mu}{\theta}\right); \quad \theta > 0, \mu > 0.$$

Thus

$$f(x|\mu,\sigma)g(\mu) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left[-\frac{1}{2}\frac{(\log_e x - \mu)^2}{\sigma^2}\right] \cdot \frac{1}{\theta} \exp\left(-\frac{\mu}{\theta}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma x} \cdot \frac{1}{\theta} \exp\left[-\frac{1}{2}\frac{(\log_e x - \mu)^2}{\sigma^2} - \frac{\mu}{\theta}\right]$$

Consider

$$h(x) = \int_0^\infty f(x|\mu)\, g(\mu)\, d\mu$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma x} \cdot \frac{1}{\theta} \exp\left[ -\frac{(\log_e x - \mu)^2}{2\sigma^2} - \frac{\mu}{\theta} \right] d\mu$$

Let

$$y = \frac{\mu - \log_e x + \sigma^2/\theta}{\sigma}, \ \mu = \sigma y + \log_e x - \sigma^2/\theta$$

$$\mu = 0 \rightarrow y = \frac{-\log_e x + \sigma^2/\theta}{\sigma}$$

$$\mu = \infty \rightarrow y = \infty$$

$$\frac{d\mu}{dy} = \sigma \rightarrow d\mu = \sigma\, dy$$

$$h(x) = \frac{1}{x} \cdot \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\theta} \int_{\frac{-\log x + \sigma^2/\theta}{\sigma}}^{\infty} \exp\left[ -\frac{(\log x - \sigma y - \log x + \sigma^2/\theta)^2}{2\sigma^2} - \frac{\sigma y + \log x - \sigma^2/\theta}{\theta} \right] \sigma\, dy$$

$$= \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\theta} \int_{\frac{-\log x + \sigma^2/\theta}{\sigma}}^{\infty} \exp\left[ -\frac{(\sigma^2/\theta - \sigma y)^2}{2\sigma^2} - \frac{\sigma y + \log x - \sigma^2/\theta}{\theta} \right] dy$$

$$= \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\theta} \int_{\frac{-\log x + \sigma^2/\theta}{\sigma}}^{\infty} \exp\left[ -\frac{(\sigma^4/\theta^2 - 2\sigma^3 y/\theta + \sigma^2 y^2)}{2\sigma^2} - \frac{\sigma y + \log x - \sigma^2/\theta}{\theta} \right] dy$$

$$= \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\theta}$$

$$\int_{\frac{-\log x + \sigma^2/\theta}{\sigma}}^{\infty} \exp\left[ \frac{-\theta(\sigma^4/\theta^2 - 2\sigma^3 y/\theta + \sigma^2 y^2) - 2\sigma^2(\sigma y + \log x - \sigma^2/\theta)}{2\theta\sigma^2} \right] dy$$

$$= \frac{1}{x}\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\theta}$$

$$\int_{\frac{-\log x+\sigma^2/\theta}{\sigma}}^{\infty} \exp\left[\frac{-\sigma^4/\theta+2\sigma^3 y-\theta\sigma^2 y^2-2\sigma^3 y-2\sigma^2\log x+2\sigma^4/\theta}{2\theta\sigma^2}\right]dy$$

$$= \frac{1}{x}\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{\theta}\int_{\frac{-\log x+\sigma^2/\theta}{\sigma}}^{\infty}\exp\left[\frac{-\theta\sigma^2 y^2}{2\theta\sigma^2}\right]dy\exp\left[-\frac{\sigma^4}{2\theta^2\sigma^2}-\frac{2\sigma^2\log x}{2\theta\sigma^2}+\frac{2\sigma^4}{2\theta^2\sigma^2}\right]$$

$$= \frac{1}{x}\cdot\frac{1}{\theta}\cdot\frac{1}{\sqrt{2\pi}}\int_{\frac{-\log x+\sigma^2/\theta}{\sigma}}^{\infty}\exp\left[\frac{-y^2}{2}\right]dy\exp\left[\frac{\sigma^2}{2\theta^2}-\frac{\log x}{\theta}\right]$$

$$= \frac{1}{x}\cdot\frac{1}{\theta}\exp\left[\frac{\sigma^2}{2\theta^2}-\frac{\log x}{\theta}\right]\left[\Phi(\infty)-\Phi\left(\frac{-\log x+\sigma^2/\theta}{\sigma}\right)\right]$$

$$= \frac{1}{x}\cdot\frac{1}{\theta}\exp\left(\frac{\sigma^2}{2\theta^2}\right)x^{-\frac{1}{\theta}}\left[1-\Phi\left(\frac{-\log x+\sigma^2/\theta}{\sigma}\right)\right]$$

$$= \frac{x^{-\left(1+\frac{1}{\theta}\right)}}{\theta}\exp\left(\frac{\sigma^2}{2\theta^2}\right)\left[1-\Phi\left(\frac{-\log x+\sigma^2/\theta}{\sigma}\right)\right]$$

Thus, the pdf of Lognormal-Exponential distribution $LN-E(\sigma,\theta)$ is

$$h(x) = \frac{x^{-\left(1+\frac{1}{\theta}\right)}}{\theta}\exp\left(\frac{\sigma^2}{2\theta^2}\right)\left[1-\Phi\left(\frac{-\log x+\sigma^2/\theta}{\sigma}\right)\right].$$

## E.2 Laplace-Gamma Distribution

**Laplace distribution**

A random variable $X$ is said to be Laplace distributed with parameter $\mu$ and $\beta$ denoted by $X \sim LN(\mu,\beta)$.

$$\text{PDF} \quad : \quad f_X\left(x\middle|\mu,\beta\right) = \frac{1}{2\beta}\exp\left[-\frac{|x-\mu|}{\beta}\right]; \; \mu \in R, \beta > 0, x \in R.$$

Case1. $x-\mu \quad , \quad x > \mu.$

Case2. $\mu - x \quad , \quad x < \mu.$

**Gamma distribution**

A random variable $\Theta$ is said to be Gamma distributed with parameter $\alpha$ and $\sigma$ denoted by $\Theta \sim Gam(\alpha,\sigma)$.

$$\text{PDF} \quad : \quad g(\mu) = \frac{\sigma^{\alpha}}{\Gamma(\alpha)}\exp[-\sigma\mu]\mu^{\alpha-1}; \quad \sigma,\alpha > 0, \mu > 0.$$

Hence;

*Case1.* $x-\mu \quad , \quad x > \mu$

$$h(x) = \int_0^\infty \frac{1}{2\beta}\exp\left[-\frac{x-\mu}{\beta}\right]\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\exp[-\sigma\mu]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[-\frac{x-\mu}{\beta}\right]\exp[-\sigma\mu]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[\frac{-x-\mu}{\beta}-\sigma\mu\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[\frac{-x-\mu-\sigma\beta\mu}{\beta}\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[\frac{-x}{\beta}\right]\exp\left[\frac{-\mu-\sigma\beta\mu}{\beta}\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\exp\left[\frac{-x}{\beta}\right]\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[\frac{-\mu-\sigma\beta\mu}{\beta}\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\exp\left[\frac{-x}{\beta}\right]\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_0^\infty \exp\left[-\mu\left(\frac{1+\sigma\beta}{\beta}\right)\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\exp\left[\frac{-x}{\beta}\right]\sigma^{\alpha}\int_{0}^{\infty}\frac{1}{\Gamma(\alpha)}\left(\frac{1+\sigma\beta}{\beta}\right)^{\alpha}\exp\left[-\mu\left(\frac{1+\sigma\beta}{\beta}\right)\right]\mu^{\alpha-1}\,d\mu\cdot\left(\frac{\beta}{1+\sigma\beta}\right)^{\alpha}$$

Compare Gamma (pdf) with parameter $\alpha \;\; and \;\; \left(\frac{1+\sigma\beta}{\beta}\right)$

$$= \frac{1}{2\beta}\sigma^{\alpha}\left(\frac{\beta}{1+\sigma\beta}\right)^{\alpha}\exp\left[\frac{-x}{\beta}\right]$$

$$= \frac{1}{2\beta}\left(\frac{\sigma\beta}{1+\sigma\beta}\right)^{\alpha}\exp\left[\frac{-x}{\beta}\right]$$

Thus, the pdf of Laplace-Gamma distribution $La-G(\sigma,\beta,\alpha)$ is

$$h(x)=\frac{1}{2\beta}\left(\frac{\sigma\beta}{1+\sigma\beta}\right)^{\alpha}\exp\left[\frac{-x}{\beta}\right].$$

*Case2.* $\mu-x \quad , \quad x<\mu$

$$h(x) = \int_{0}^{\infty}\frac{1}{2\beta}\exp\left[-\frac{\mu-x}{\beta}\right]\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\exp[-\sigma\mu]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}\exp\left[-\frac{\mu-x}{\beta}\right]\exp[-\sigma\mu]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}\exp\left[-\frac{\mu-x}{\beta}-\sigma\mu\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}\exp\left[\frac{-\mu-x-\sigma\beta\mu}{\beta}\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\frac{\sigma^{\alpha}}{\Gamma(\alpha)}\int_{0}^{\infty}\exp\left[-\frac{x}{\beta}\right]\exp\left[\frac{-\mu-\sigma\beta\mu}{\beta}\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\exp\left[-\frac{x}{\beta}\right]\sigma^{\alpha}\int_{0}^{\infty}\frac{1}{\Gamma(\alpha)}\exp\left[-\mu\left(\frac{\sigma\beta+1}{\beta}\right)\right]\mu^{\alpha-1}\,d\mu$$

$$= \frac{1}{2\beta}\exp\left[-\frac{x}{\beta}\right]\sigma^{\alpha}\int_{0}^{\infty}\frac{1}{\Gamma(\alpha)}\left(\frac{\sigma\beta+1}{\beta}\right)^{\alpha}\exp\left[-\mu\left(\frac{\sigma\beta+1}{\beta}\right)\right]\mu^{\alpha-1}\,d\mu\cdot\left(\frac{\beta}{\sigma\beta+1}\right)^{\alpha}$$

Compare Gamma (pdf) with parameter $\alpha$ *and* $\left(\frac{\sigma\beta+1}{\beta}\right)$

$$= \frac{1}{2\beta}\,\sigma^{\alpha}\left(\frac{\beta}{\sigma\beta+1}\right)^{\alpha}\exp\left[-\frac{x}{\beta}\right]$$

$$= \frac{1}{2\beta}\left(\frac{\sigma\beta}{\sigma\beta+1}\right)^{\alpha}\exp\left[-\frac{x}{\beta}\right]$$

Thus, the pdf of Laplace-Gamma distribution $La-G(\sigma,\beta,\alpha)$ is

$$h(x) = \frac{1}{2\beta}\left(\frac{\sigma\beta}{\sigma\beta+1}\right)^{\alpha}\exp\left[-\frac{x}{\beta}\right].$$

## E.3 Exponential – Erlang Distribution

**Exponential distribution**

A random variable $X$ is said to be Exponentially distributed with parameter $\mu$ denoted by $X \sim Exp(\mu)$.

$$\text{PDF} \quad : \quad f_X(x|\mu) = \mu e^{-\mu x}; \quad \mu > 0, x > 0.$$

**Erlang distribution**

A random variable $\Theta$ is said to be Erlang distributed with parameter $k$ and $\lambda$ denoted by $\Theta \sim Erlang(k,\lambda)$.

$$\text{PDF} \quad : \quad g(\mu) = \frac{\lambda^{k}\mu^{k-1}}{(k-1)!}e^{(-\lambda\mu)}, \quad k > 0,\ k \text{ is integer},\ \lambda > 0, \mu > 0.$$

Consider

$$h(x) = \int_{0}^{\infty} f(x|\mu)\, g(\mu)\, d\mu$$

$$= \int_0^\infty \mu\, \mathrm{e}^{(-\mu x)} \cdot \frac{\lambda^k \mu^{k-1}\, \mathrm{e}^{(-\lambda\mu)}}{(k-1)!} d\mu$$

$$= \int_0^\infty \frac{\lambda^k \mu^{k-1-1}\, \mathrm{e}^{(-\mu x-\lambda\mu)}}{(k-1)!} d\mu$$

$$= \int_0^\infty \frac{(x+\lambda)^{k+1}\, \mu^{(k+1)-1} e^{-\mu(x+\lambda)}}{(k+1-1)!} d\mu \cdot \frac{\lambda^k}{(k-1)!} \cdot \frac{k!}{(x+\lambda)^{k+1}}$$

$$= \frac{\lambda^k}{(k-1)!} \cdot \frac{k!}{(x+\lambda)^{k+1}} = \frac{\lambda^k k}{(x+\lambda)^{k+1}}.$$

Thus, the pdf of Exponential-Erlang distribution $E-Erlang(k,\lambda)$ is $\dfrac{\lambda^k k}{(x+\lambda)^{k+1}}$.

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