MINIMUM INITIAL CAPITAL AND VALUE FUNCTION PROBLEMS IN INSURANCE

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Suranaree University of Technology has approved this thesis submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

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SURPLUS PROCESS/ RUIN PROBABILITY/ MINIMUM INITIAL CAPITAL/ VALUE FUNCTION/ REINSURANCE/ INVESTMENT

The study in this thesis is in the framework of the discrete-time risk model (or surplus process) in insurance and is separated into two parts.

In the first part, the discrete-time surplus process is studied under the regulation that the insurance company has to reserve sufficient initial capital to ensure that ruin probability does not exceed the given quantity α. The process is considered in the situation that the possible insolvency can occur only at claim arrival times $T_n = n, n = 1, 2, 3, \cdots$. We study the relationship between the initial capital and the ruin probability, and prove the existence of the minimum initial capital. Moreover, we give an example in approximating the minimum initial capital in the case of exponential claims.

In the second part, the discrete-time surplus process is studied in situation that the surplus process can be controlled by two activities; one is reinsurance for which the reinsurance company has an opportunity to default and the other is an investment in a financial market. We prove the existence of an optimal plan and derive a formula for the value function. Finally, approximating the optimal retention level in the case of proportional reinsurance is given as an example.

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Watcharin Klongdee

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CHAPTER I

INTRODUCTION

Financial risk management is traditionally separated into market risk and credit risk. Market risk is the risk due to the fluctuations of market variables and is the best known type of risk in banking. It is the risk of a change in value of a financial position due to changes in the value of the underlying components on which that position depends, such as cash products (stock, bond), derivatives (plain vanilla, exotics), interest rate, equities, foreign exchange rate, emerging markets, commodities, etc. Each component will have its own risk management expertise requirement. Credit risk, or default risk, is the possibility that a borrower will be unable to repay principal and interest as agreed in the loan repayment contract. In the U.S., default risk is estimated by a credit rating from Standard & Poor's, Moody's or some other rating agency. Investors control default risk by monitoring the ratings of the bonds they hold or consider for purchase.

At present, the insurance industry in many countries over the world has grown at a faster pace, so the insurance risk management problem arises consequently. Insurance risk, concerning actuarial science, is considered in addition to market risk and credit risk. The main problem is, how the insurance company can manage the capital reserve for customer compensation according to its liabilities. This means that the insurance company has the risk of the insolvency possibility when its surplus becomes negative. Therefore, risk models have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserve of the insurance company.

1.1 Classical Risk Model

In 1903 the Swedish actuary Filip Lundberg laid the foundations of modern risk theory. Risk theory is a synonym for non-life insurance mathematics, which deals with the modeling of claims that arrive in an insurance business and which gives advice on how much premium has to be charged in order to avoid insolvency of the insurance company.

One of Lundberg's main contributions is the introduction of a simple model which is capable of describing the basic dynamics of a homogeneous insurance portfolio. By this we mean a portfolio of contracts or policies for similar risks such as automobile insurance for a particular kind of car, insurance against theft in households or insurance against water damage of one-family homes. There are three assumptions in the model

- Claims happen at the times T_i satisfying

  $$0 = T_0 \leq T_1 \leq T_2 \leq \cdots .$$

  We called them *claim arrivals* or *claim times* or *claim arrival times*. $\{T_n, n \in \mathbb{N}_0\}$ is called a *claim arrival process*.

- The ith claim arriving at time T_i causes the *claim size* of claim severity Y_i. The sequence of $\{Y_n, n \in \mathbb{N}\}$ constitutes an independent and identically distributed (i.i.d.) sequence of non-negative random variables. $\{Y_n, n \in \mathbb{N}\}$ is called a *claim size process*.

- The claim size process $\{Y_n, n \in \mathbb{N}\}$ and the claim arrival process $\{T_n, n \in \mathbb{N}_0\}$ are mutually independent.

Now we can define the claim number process

$$N(t) = \max\{i \geq 1 : T_i \leq t\},$$

i.e., $\{N(t), t \geq 0\}$ is a counting process on $[0, \infty)$: $N(t)$ is the number of the claims which have occurred by time t. The object of main interest from the point of view of an insurance company is the *total claim size process*:

$$\Lambda(t) = \sum_{i=1}^{N(t)} Y_i, t \geq 0.$$

Later in the 1930s, Harald Cramér, the famous Swedish statistician and probabilist, extensively developed collective risk theory by using the total claim size process $\Lambda(t)$ with claim arrival times T_i which are generated by a Poisson process. The homogeneous Poisson process plays a major role in insurance mathematics. If we specify the claim number process as a homogeneous Poisson process the resulting model which combines claim sizes and claim arrival times is called *Cramér-Lundberg model.*

Let $p(t)$ denote the premium income in the time interval $[0, t]$. In the Cramér-Lundberg model it is assumed that $p(\cdot)$ is a deterministic linear function: that is, $p(t) = c_0 t, t \geq 0$ where $c_0 > 0$ is a constant called the *premium rate.* With the total claim amount $\Lambda(t)$, put for $t \geq 0$,

$$X(t) = x + p(t) - \Lambda(t) = x + c_0 t - \sum_{i=t}^{N(t)} Y_i. \tag{1.1}$$

The process $\{X(t), t \geq 0\}$ is called the *risk process* (or *surplus process*) of the model; here $x \geq 0$ is the initial capital. A *classical risk measure* is the infinite time ruin probability of the surplus process (1.1) which is defined by

$$\Phi(x) = \mathrm{P}(T < \infty) \tag{1.2}$$

for $X(0) = x \geq 0$ which is a function of the initial capital x where

$$T = \inf\{t \geq 0, X(t) < 0\}. \tag{1.3}$$

Such a T is called the *ruin time*: the first time the surplus falls below zero. Note that $\Phi(x)$ depends on the premium rate c_0 as well. Instead of the ruin probability, Gerber and Shiu (1998) introduced the quantity

$$\mathrm{E}[e^{-\delta T}\mathbb{I}_{(T<\infty)}], \delta > 0 \tag{1.4}$$

known as the *Gerber-Shiu discounted penalty function* and is a general case of (1.2).

1.2 Outline of this Thesis

To attain the major objective, we give a brief outline of how we intend to proceed and what each chapter contains. The thesis is organized as follows.

In Chapter II, we introduce some notation, terminology and some mathematical tools which are used in the main theorems.

In Chapter III, we study the discrete-time surplus process with the claim arrival times $T_n = n, n \geq 0$ according to the risk regulation. We prove the existence of the minimum initial capital and apply the bisection method to approximate the minimum initial capital for exponential claims.

In Chapter IV, we study the discrete-time surplus process which can be controlled by two activities, one is reinsurance for which the reinsurance company has an opportunity to default and the other is an investment in a financial market. We prove the existence of an optimal plan and also derive a formula for the value function. Finally, we approximate the value function and optimal plan for some well known distributions.

The conclusion of the thesis is presented in the last chapter.

CHAPTER II

PRELIMINARY

In this chapter, we introduce some terminology of insurance, the description of the Risk-Based Capital Framework and the research problems.

2.1 Insurance and Reinsurance

In law and economics, *insurance* is a form of risk management primarily used to hedge against the risk of a contingent, uncertain loss. Insurance is defined as the equitable transfer of the risk of a loss, from one entity to another, in exchange for payment. An *insurer* is a company selling the insurance; an *insured* or *policyholder* is the person or entity buying the insurance policy. The insurance rate is a factor used to determine the amount to be charged for a certain amount of insurance coverage, called the *premium*.

Reinsurance is insurance that is purchased by an insurance company (insurer) from a reinsurer as a means of risk management, to transfer risk from the insurer to the reinsurer.

Reinsurance treaties are mutual agreements between different insurance companies with the aim to reduce the risk in a particular insurance portfolio by sharing the risk of the occurring claims as well as the premium in this portfolio. There are many types of reinsurance treaties, the insurance company handles two treaties in mostly as follows:

- *Proportional reinsurance.* In a proportional reinsurance treaties each individual claim of size Y is divided between insurer and reinsurer to a proportionality factor $b \in [0, 1]$: the insurer pays bY, the reinsurer pays $(1 - b)Y$.

- *Excess-of-loss reinsurance.* In excess of loss (XL) reinsurance each claim of size Y is divided between the insurer and the reinsurer according to priority $b \in [0, \infty]$: the insurer pays $\min\{Y, b\}$ and the reinsurer pays $\max\{Y - b, 0\}$.

The b from above is called the *ceding company's retention level* or *retention level.*

2.2 Risk-Based Capital (RBC) Framework

A Risk Based Capital Framework (the RBC Framework) specifies the capital which an insurer needs to have in excess of its liabilities (mostly technical reserves) based on the risk profile of the individual insurer. The difference between this approach and the current formula that is in place in Thailand is that:

- The required capital depends on the risk profile of the specific insurer, rather than just the size of its business.

- The basis for valuation of assets and liabilities is more transparent than before, and no longer incorporates undisclosed margins.

The RBC Framework seeks to amend the existing valuation methodology for assets and liabilities, establish new capital requirement rules, update the role of actuaries, introduce a new set of statutory reporting standards and introduce new regulations or review existing regulations and consider how and when these may require to be amended to harmonise with the new framework. The framework will set out draft regulations, notices and guidelines (including actuarial guidelines).

2.2.1 Aims of Risk Based Capital in Thailand

The objective of Solvency Capital is to provide a buffer to protect the interests of policyholders. This buffer should be sufficiently large to allow time for management action or regulator action to counter the impact of adverse experience on the ability of the insurer to meet its liabilities to customers. The proposed new basis for Solvency Capital is a Risk Based Capital Framework (RBC Framework).

That is, the required Solvency Capital will directly reflect the risks to which an individual company is exposed. RBC would replace the existing one size fits all system.

The RBC Framework specifies the capital which an insurer needs to have in excess of its liabilities (mostly technical reserves). The difference between the proposed new approach and the current formula is that:

- The required capital depends on the risk profile of the specific insurer, rather than just the size of its business; and
- The basis for valuation of assets and liabilities is more transparent than before, and no longer incorporates undisclosed margins. Such a Framework will drive better risk management and more efficient use of capital.

The proposed RBC Framework itself will trigger changes to regulation in a number of areas. In addition, the role of actuaries and auditors will change to reflect the increased weight given to an insurers own situation in assessing its Solvency Capital requirement.

The RBC Framework has been designed specifically for Thailand, recognizing and reflecting the current situation of the whole industry. In particular, the process recognized that the size and capabilities of different insurers are different, and this is reflected in the relatively standardized and straightforward approach which is proposed.

Where appropriate, certain features of international best practice, drawn in particular from Solvency II and the Malaysian and Singaporean regimes have been incorporated in the proposed RBC Framework. However, the circumstances and interests of Thai companies have always been uppermost in the designers minds.

2.2.2 Principles of Risk Based Capital and Applicability

All insurers and reinsurers, including branches, will be subject to the RBC requirements. The following principles underlie the proposed RBC Framework:

- Allow insurers greater opportunity to achieve efficient use of their capital by linking the required capital more closely to the level of risk entailed by the chosen business strategy.
- Aligned, where possible, with international best practice.
- Provide the regulator with relevant and timely information within the context of specific risk capital levels, to provide adequate early warning for timely intervention.
- Consistent between life and non-life companies.
- Capital Requirement varies with risk and scale: the capital required of two insurers with similar liabilities and similar risks is to be consistent; conversely, two insurers whose risk profiles are significantly different should experience corresponding capital requirements.
- Thai Government guarantee to be considered the highest level of security (i.e., risk free).

- Separation of buffers and margins from the estimates of technical reserves, combined with explicit levels for the technical reserves, to allow greater transparency and greater comparability of insurers solvency positions.

Some insurers may have developed or have access to internal models for setting their own capital targets or for the purposes of reporting under the Solvency II regime. While the Office of Insurance Commission (OIC) wishes to encourage the development of such models, all companies must still comply with the standard RBC Framework and the results from internal models may not be used as a substitute at present. The OIC will consider allowing the use of such models in the future.

2.2.3 The Formula for the Capital Requirement

The Capital Adequacy Ratio (CAR) for an insurer is defined as: (Total Available Capital) divided by (Risk Capital Requirement). The RBC Framework sets the target to be met by an insurer as a Capital Adequacy Ratio at least equal to 100%. This means that

$$\text{CAR} = \frac{\text{Total Available Capital}}{\text{Risk Capital Requirment}} \geq 100\%.$$

The risks to be taken into account in the Risk Capital requirement are grouped into the following categories:

1. *Group risk* represents the risks associated with membership of a wider business grouping such that risks to which other group companies are exposed could have a financial or operational impact on the insurer.

2. *Operational risk* is produced by inadequate or failing internal processes, persons or systems, or by external events. Examples are fraud or liability for mis-selling.

3. *Liquidity risk* describes the risk that an insurer while balance sheet solvent, cannot generate enough cash to pay claims and other outgoings.

4. *Market risk* derives from market prices themselves or from the volatility of those market prices. Among other risks, market risk includes equity risk, interest rate risk, property risk and currency risk.

5. *Credit risk* includes both the risk that issuer of a bond, or other creditor defaults and the risk that the counterparty in a risk mitigating contract is unable to meet its obligations to the insurer. This latter risk is especially relevant to reinsurance contracts, but also arises in the case of financial derivatives.

6. *Life insurance liability risk* is the risk specific to policies held with life insurers. It includes the risk of unexpectedly high or low mortality or morbidity among policyholders, or an unexpectedly large increase in administrative costs.

7. *Non-life insurance liability risk* is the risk specific to non-life insurance policies associated in particular with unexpectedly many or unexpectedly high claims.

The Risk Capital Requirement is calculated by applying risk charges to the value of specific items within the assets and the liabilities and to exposure measures for other risks. Each type of asset or liability attracts a charge according to the risks to which it is exposed and the sum (subject to diversification adjustments) of these charges equals the Risk Capital Requirement.

There will continue to be an absolute minimum amount of capital which an insurer or reinsurer must hold. This amount is currently Baht 50m for life insurers and Baht 30m for general insurers. These amounts will be reviewed once the RBC Framework is completed.

The RBC Framework is not a substitute for good risk management, but acts to strengthen it. Insurers are expected to continue to develop and implement sound risk management and governance regimes. Insurers will be required to make sure that their strategy, internal controls and decision making processes are effective in ensuring that they assume only their intended level of risk. In addition, insurers will be expected to actively manage their capital adequacy ratio by taking into account the potential impact of business strategies on the insurers risk profile as part of the decision making process. The OIC will retain the power to intervene in the management of companies which do not meet adequate risk management standards.

2.2.4 RBC Framework of Singapore and Malaysia

For the non-life insurance in Singapore and Malaysia, they have the regulatory control level 120% and 130%, respectively, i.e.,

$$\begin{aligned}\text{Capital Adequacy Ratio (CAR)} &= \frac{\text{Total Capital Available (TCA)}}{\text{Total Capital Requirement (TCR)}} \\ &\geq 120\%, 130\%, \text{respectively},\end{aligned}$$

where

$$\begin{aligned}\text{TCR} &= \text{Insurance Risk Capital Charge+Market Risk Capital Charge} \\ &\quad \text{+Credit Risk Capital Charge.}\end{aligned}$$

2.3 Relationship between RBC Framework and Research Problems

We recall the surplus process $\{X(t), t \geq 0\}$ as mentioned in (1.1), i.e.,

$$X(t) = x + c_0 t - \sum_{i=1}^{N(t)} Y_i$$

where c_0 is the premium rate for one unit timeand $\{Y_n, n \in \mathbb{N}\}$ is the claim size process. Since the nth claim arrives at the time T_n, the possible insolvency can occur only at claim arrival times $T_n, n \in \mathbb{N}$. Thus, we are only interested in the surplus at time T_n. Since $N(T_n) = n$, the surplus at time T_n equals to

$$X_n = x + c_0 T_n - \sum_{i=1}^{n} Y_i \tag{2.1}$$

where $X_n := X(T_n)$ for all $n \in \mathbb{N}_0$ and $X_0 := X(0) = x$ is the initial capital. Then, $\{X_n, n \in \mathbb{N}_0\}$ as mentioned in (2.1) is called the *discrete-time surplus process.*

We say that the surplus process such that the inter-arrival process $\{Z_n = T_n - T_{n-1}, n \in \mathbb{N}\}$ and the claim size process $\{Y_n, n \in \mathbb{N}\}$ are i.i.d., satisfying the *net profit condition* (*NPC*), if

$$\mathrm{E}[Y_1] - c_0 \mathrm{E}[T_1] < 0. \tag{2.2}$$

The interpretation of the NPC is rather intuitive. In a given unit of time the expected claim size $\mathrm{E}[X_1]$ has to be smaller than the premium income in this unit time, represented by the expected premium $c_0 \mathrm{E}[T_1]$ when c_0 is a premium rate for one unit time. From inequality (2.2), there exists $\theta_0 > 0$ such that

$$c_0 = (1 + \theta_0) \frac{\mathrm{E}[Y_1]}{\mathrm{E}[T_1]}, \tag{2.3}$$

which is called the *expected value premium principle.* The quantity θ_0 is said to be the *safety loading* of insurer.

In this thesis, we consider the two different problems of the discrete-time surplus process (2.1). For the first process, we consider the surplus process (2.1) in the situation that the possible insolvency can occur only at claim arrival times $T_n = n$, $n \in \mathbb{N}$. We consider the ruin probability-based initial capital problem (or minimum initial capital problem) as the RBC problem, i.e., we can consider the ruin probability as the insurance risk and the initial capital as the insurance risk capital charge. For the second process, we consider the discrete-time surplus process with the claim arrival process $\{T_n, n \in \mathbb{N}_0\}$ which is a stochastic process under the investment and reinsurance credit risk. The activities, making the insurance risk, market risk and (reinsurance) credit risk, can be included in this process.

CHAPTER III

MINIMUM INITIAL CAPITAL PROBLEM

In this chapter, we study the minimum initial capital problem of the discrete-time surplus process under the claim arrival times $T_n = n, n \in \mathbb{N}$ and we consider the relationship between ruin probability and initial capital.

3.1 Model Descriptions

We consider the discrete-time surplus process in the situation that the possible insolvency can occur only at claim arrival times $T_n = n, n \in \mathbb{N}$. The nth claim arriving at time n causes the claim size Y_n. Let the positive random variable Y_n be the claim size at time n defined in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ for all $n \in \mathbb{N}$. process. We assume that $Y_n, n \in \mathbb{N}$ are independent and identically distributed (i.i.d.) random variables, i.e., $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process. Now let the constant $c_0 > 0$ represent the premium rate for one unit time which is calculated by the expected value premium principle, i.e.,

$$c_0 = (1 + \theta_0)E(Y_1) \tag{3.1}$$

where $\theta_0 > 0$ is the safety loading of the insurer. Thus, the quantity $c_0 n$ describes the inflow of capital into the business by time n, and the random variable $\sum_{i=1}^{n} Y_i$ describes the outflow of capital due to payments for claims occurring in $\{1, 2, 3, \cdots, n\}$. Therefore, the quantity

$$X_n = x + c_0 n - \sum_{i=1}^{n} Y_i \tag{3.2}$$

is the surplus at time n with the constant $X_0 = x \geq 0$ as the initial capital.

The general approach for studying ruin probability in the discrete-time surplus process is through the so-called *Gerber-Shiu discounted penalty function.* This approach has appeared in Pavlova and Willmot (2004), Dickson (2005) and Li (2005a,b). In these articles, they studied the ruin probability as a function of the initial capital $x \geq 0$. In this chapter, we shall work in the opposite direction, i.e., we study the initial capital for discrete-time surplus process as a function of ruin probability.

3.2 Ruin Probability Behaviors

Let $\{X_n, n \in \mathbb{N}_0\}$ be the discrete-time surplus process as in section 3.1. We consider the finite-time ruin probabilities of the surplus process $\{X_n, n \in \mathbb{N}_0\}$ which is driven by the i.i.d. claim size process $\{Y_n, n \in \mathbb{N}\}$ and the premium rate $c_0 > 0$.

Let $x \geq 0$ be an initial capital. For each $n \in \mathbb{N}$, we let

$$\varphi_n(x) \quad := \quad \mathrm{P}(X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, \cdots, X_n \geq 0 | X_0 = x) \tag{3.3}$$

denote the *survival probability* at the times n. Thus, the *ruin probability* at one of the times $1, 2, 3, \cdots, n$ is denoted by

$$\begin{aligned} \Phi_n(x) &= 1 - \varphi_n(x) \\ &= \mathrm{P}(X_i < 0 \text{ for some } i \in \{1, 2, 3, \cdots, n\} | X_0 = x). \end{aligned} \tag{3.4}$$

Definition 3.1. Let $\{X_n, n \in \mathbb{N}_0\}$ be a surplus process which is driven by the claim size process $\{Y_n, n \in \mathbb{N}\}$ and the premium rate $c_0 > 0$. Let $\alpha \in (0, 1)$ and let $N \in \mathbb{N}$ be given. Let $x \geq 0$ be an initial capital. If $\Phi_N(x) \leq \alpha$, then x is called an *acceptable initial capital* corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$. Particularly, if

$$x^* = \min_{x \geq 0} \{x : \Phi_N(x) \leq \alpha\} \tag{3.5}$$

exists, then x^* is called the *minimum initial capital* corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ and is written as

$$x^* := \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}). \tag{3.6}$$

3.2.1 Ruin and Survival Probability

We define the *total claim size process* $\{\Lambda_n, n \in \mathbb{N}\}$ by

$$\Lambda_n := Y_1 + Y_2 + Y_3 + \cdots + Y_n, n \in \mathbb{N}. \tag{3.7}$$

The survival probability at the time N as mentioned in (3.3) can be expressed as follows:

$$\begin{aligned} \varphi_N(x) &= \mathrm{P}\big(\Lambda_1 \le x + c_0, \Lambda_2 \le x + 2c_0, \cdots, \Lambda_N \le x + Nc_0\big) \\ &= \mathrm{P}\Big(\bigcap_{k=1}^{N} \{\omega : \Lambda_k(\omega) \le x + kc_0\}\Big). \end{aligned} \tag{3.8}$$

From equation (3.8), we have

$$\varphi_N(x) = \mathrm{E}\left[\prod_{k=1}^{N} \mathbb{I}_{(-\infty,0]}\left(\Lambda_k - kc_0 - x\right)\right] \tag{3.9}$$

where

$$\mathbb{I}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A \end{cases}$$

for all $A \subset \mathbb{R}$. For each $a \in \mathbb{R}$ and $x \ge 0$, we obtain

$$\mathbb{I}_{(-\infty,0]}(a - x) = \begin{cases} 1, & a \le x, \\ 0, & x < a. \end{cases}$$

Then $\mathbb{I}_{(-\infty,0]}(a - x)$ is increasing and right continuous in x. This implies that $\prod_{k=1}^{N} \mathbb{I}_{(-\infty,0]}(a_k - x)$ is also increasing and right continuous in x where $a_k \in \mathbb{R}$, $k = 1, 2, 3, \cdots, N$. By the Lebesgue's dominated convergence theorem (Theorem

C.2), we have

$$\begin{aligned}
\lim_{\nu\to x^+}\varphi_N(\nu) &= \lim_{\nu\to x^+}\mathrm{E}\left[\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}\left(\Lambda_k-kc_0-\nu\right)\right]\\
&= \mathrm{E}\left[\lim_{\nu\to x^+}\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}\left(\Lambda_k-kc_0-\nu\right)\right]\\
&= \mathrm{E}\left[\prod_{k=1}^{N}\mathbb{I}_{(-\infty,0]}\left(\Lambda_k-kc_0-x\right)\right]\\
&= \varphi_N(x). \qquad (3.10)
\end{aligned}$$

Therefore, $\varphi_N(x)$ is increasing and right continuous. Moreover, we can conclude that $\Phi_N(x)=1-\varphi_N(x)$ is decreasing and also right continuous. Let $F_{Y_1}(y)$ be the distribution function of Y_1, i.e.,

$$F_{Y_1}(y)=\mathrm{P}(Y_1\le y)$$

for all $y\in\mathbb{R}$. Since the claim size process $\{Y_n,n\in\mathbb{N}\}$ is i.i.d., we obtain

$$F_{Y_n}(y)=F_{Y_1}(y)$$

for all $y\in\mathbb{R}$ and $n\in\mathbb{N}$.

Theorem 3.1. *Let $N\in\mathbb{N}$ and $c_0>0$ be given. If $\{Y_n,n\in\mathbb{N}\}$ is an i.i.d. claim size process, then*

$$\lim_{x\to\infty}\varphi_N(x)=1 \text{ and } \lim_{x\to\infty}\Phi_N(x)=0. \qquad (3.11)$$

Proof. Firstly, we will show that the following property holds:

$$\bigcap_{k=1}^{N}\{\omega:Y_k(\omega)\le x+c_0\}\subset\bigcap_{k=1}^{N}\left\{\omega:\Lambda_k(\omega)\le Nx+kc_0\right\}. \qquad (3.12)$$

Let $\omega_0\in\bigcap_{k=1}^{N}\{\omega:Y_k(\omega)\le x+c_0\}$ be given. For each $k\in\{1,2,3,\cdots,N\}$, we have $Y_k(\omega_0)\le x+c_0$ and

$$\Lambda_k(\omega_0)=\sum_{i=1}^{k}Y_i(\omega_0)\le kx+kc_0\le Nx+kc_0. \qquad (3.13)$$

That is, $\omega_0 \in \{\omega : \Lambda_k(\omega) \leq Nx + kc_0\}$. Therefore, (3.12) follows. Next, since the claim size process $\{Y_n, n \in \mathbb{N}\}$ is i.i.d., we obtain

$$
\begin{aligned}
\mathrm{P}\big(\bigcap_{k=1}^{N}\{\omega : Y_k(\omega) \leq x + c_0\}\big) &= \prod_{k=1}^{N} \mathrm{P}(Y_k \leq x + c_0) \\
&= \prod_{k=1}^{N} F_{Y_k}(x + c_0) \\
&= (F_{Y_1}(x + c_0))^N. \qquad (3.14)
\end{aligned}
$$

By equation (3.8), we have

$$\varphi_N(Nx) = \mathrm{P}\left(\bigcap_{k=1}^{N}\left\{\omega : \Lambda_k(\omega) \leq Nx + kc_0\right\}\right). \qquad (3.15)$$

By (3.12), (3.14) and (3.15), we obtain

$$(F_{Y_1}(x + c_0))^N \leq \varphi_N(Nx) \leq 1. \qquad (3.16)$$

Since

$$\lim_{x\to\infty} (F_{Y_1}(x + c_0))^N = 1,$$

we get

$$\lim_{x\to\infty} \varphi_N(Nx) = 1.$$

Thus, we conclude that

$$\lim_{x\to\infty} \varphi_N(x) = 1,$$

and

$$\lim_{x\to\infty} \Phi_N(x) = \lim_{x\to\infty} (1 - \varphi_N(x)) = 1 - \lim_{x\to\infty} \varphi_N(x) = 0.$$

This completes the proof. $\square$

Corollary 3.2. *Let $\alpha \in (0, 1)$, $N \in \mathbb{N}$ and $c_0 > 0$ be given. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then there exists $\tilde{x} \geq 0$ such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$.*

Proof. We consider the following cases:

Case 1. $0 \leq \Phi_N(0) \leq \alpha$. Since $\Phi_N(x)$ is decreasing, we obtain

$$\Phi_N(x) \leq \Phi_N(0) \leq \alpha$$

for all $x \geq 0$.

Case 2. $\Phi_N(0) > \alpha$. By Theorem 3.1, we have $\Phi_N(x) \to 0$ as $x \to \infty$. Thus, there exists $\tilde{x} > 0$ such that $\Phi_N(\tilde{x}) < \alpha$. Since $\Phi_N(x)$ is decreasing, we conclude that

$$\Phi_N(x) \leq \Phi_N(\tilde{x}) < \alpha$$

for all $x \geq \tilde{x}$. □

3.2.2 Bound for the Ruin Probability

From Theorem 3.1 and Corollary 3.2, we know that a small ruin probability can be obtained by sufficiently large initial capital. In this part, we shall describe the upper bound of ruin probability with negative exponential. In order to prove the following lemma, we shall use an equivalent definition of the ruin probability which is given as follows:

$$\Phi_n(x) = \mathrm{P}\big(\max_{1\leq k\leq n}(\sum_{i=1}^{k} Y_i - c_0 k) > x\big). \tag{3.17}$$

Lemma 3.3. *Let $N \in \mathbb{N}$, $c_0 > 0$ and $x \geq 0$ be given. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then the ruin probability at one of the times $1, 2, 3, \cdots, N$ satisfies the following equation*

$$\Phi_N(x) = \Phi_1(x) + \int_{-\infty}^{x+c_0} \Phi_{N-1}(x + c_0 - y)dF_{Y_1}(y) \tag{3.18}$$

where $\Phi_0(x) = 0$.

Proof. We will prove (3.18) by induction. We start with $n = 1$. Since $\Phi_0(x) = 0$ for all $x \geq 0$, we have

$$\int_{-\infty}^{x+c_0} \Phi_0(x + c_0 - y) dF_{Y_1}(y) = 0. \tag{3.19}$$

This proves (3.18) for $n = 1$. Now assume that (3.18) holds for $n = k \geq 1$. Then

$$\begin{aligned}
\Phi_{k+1}(x) &= \mathrm{P}\left(\max_{1\leq n\leq k+1}(\sum_{i=1}^{n} Y_i - c_0 n) > x\right) \\
&= \Phi_1(x) + \mathrm{P}\big(\max_{2\leq n\leq k+1}(Y_1 + \sum_{i=2}^{n} Y_i - c_0 n) > x, Y_1 \leq x + c_0\big) \\
&= \Phi_1(x) + \mathrm{E}\big[\mathbb{I}_{(Y_1\leq x+c_0, \max\limits_{2\leq n\leq k+1}(Y_1+\sum_{i=2}^{n} Y_i - c_0 n)>x)}\big] \\
&= \Phi_1(x) + \mathrm{E}\big[\mathbb{I}_{(Y_1\leq x+c_0)} \cdot \mathbb{I}_{(\max\limits_{2\leq n\leq k+1}(Y_1+\sum_{i=2}^{n} Y_i - c_0 n)>x)}\big] \\
&= \Phi_1(x) + \mathrm{E}\big[\mathbb{I}_{(-\infty,x+c_0]}(Y_1) \cdot \mathbb{I}_{(x,\infty)}\big(\max_{2\leq n\leq k+1}(Y_1 + \sum_{i=2}^{n} Y_i - c_0 n)\big)\big].
\end{aligned} \tag{3.20}$$

We consider the second term of the right-hand side of (3.20). By Proposition D.1(i) and (iv), we obtain

$$\begin{aligned}
&\mathrm{E}\big[\mathbb{I}_{(-\infty,x+c_0]}(Y_1) \cdot \mathbb{I}_{(x,\infty)}\big(\max_{2\leq n\leq k+1}(Y_1 + \sum_{i=2}^{n} Y_i - c_0 n)\big)\big] \\
&\quad = \mathrm{E}\big[\mathbb{E}[\mathbb{I}_{(-\infty,x+c_0]}(Y_1) \cdot \mathbb{I}_{(x,\infty)}\big(\max_{2\leq n\leq k+1}(Y_1 + \sum_{i=2}^{n} Y_i - c_0 n)\big)|\sigma(Y_1)]\big] \\
&\quad = \mathrm{E}\big[\mathbb{I}_{(-\infty,x+c_0]}(Y_1)\mathbb{E}[\mathbb{I}_{(x,\infty)}\big(\max_{2\leq n\leq k+1}(Y_1 + \sum_{i=2}^{n} Y_i - c_0 n)\big)|\sigma(Y_1)]\big].
\end{aligned} \tag{3.21}$$

Since the claim sizes $Y_n, n \in \mathbb{N}$ are independent, we obtain that Y_1 and $\sum_{i=2}^{n} Y_i$

are also independent for all $n \in \{2,3,4,\cdots\}$. By Theorem D.5, we have

$$\begin{aligned}
&\mathbb{E}[\mathbb{I}_{(x,\infty)}(\max_{2\le n\le k+1}(Y_1(\omega)+\sum_{i=2}^{n}Y_i-c_0n))|\sigma(Y_1)] \\
&\qquad = \mathrm{E}[\mathbb{I}_{(x,\infty)}(\max_{2\le n\le k+1}(Y_1(\omega)+\sum_{i=2}^{n}Y_i-c_0n))] \quad \text{a.s.} \\
&\qquad = \mathrm{E}[\mathbb{I}_{(x-Y_1(\omega),\infty)}(\max_{2\le n\le k+1}(\sum_{i=2}^{n}Y_i-c_0n))] \\
&\qquad = \mathrm{P}(\max_{2\le n\le k+1}(\sum_{i=2}^{n}Y_i-c_0n)>x-Y_1(\omega)) \\
&\qquad = \mathrm{P}(\max_{2\le n\le k+1}(\sum_{i=2}^{n}Y_i-c_0(n-1))>x+c_0-Y_1(\omega)) \\
&\qquad = \Phi_k(x+c_0-Y_1(\omega)).
\end{aligned} \tag{3.22}$$

By combining (3.20), (3.21) and (3.22), we have

$$\begin{aligned}
\Phi_{k+1}(x) &= \Phi_1(x)+\mathrm{E}\big[\mathbb{I}_{(-\infty,x+c_0]}(Y_1)\cdot\Phi_k(x+c_0-Y_1)\big] \\
&= \Phi_1(x)+\int_{Y_1^{-1}(-\infty,x+c_0]}\Phi_k(x+c_0-Y_1)d\mathrm{P} \\
&= \Phi_1(x)+\int_{-\infty}^{x+c_0}\Phi_k(x+c_0-y)dF_{Y_1}(y),
\end{aligned} \tag{3.23}$$

which proves (3.18) for $n=k+1$ and concludes the proof. □

Remark 3.1. Let $N \in \mathbb{N}$ and $x \ge 0$ be given. Assume that $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. exponential claim size process with intensity $\lambda > 0$, i.e., Y_1 has the probability density function

$$f_{Y_1}(y) = \lambda e^{-\lambda y}.$$

The obtained ruin probability is in the following recursive form

$$\Phi_N(x) = \Phi_{N-1}(x) + \frac{[\lambda(x+Nc_0)]^{N-1}}{(N-1)!}e^{-\lambda(x+Nc_0)}\frac{x+c_0}{x+Nc_0} \tag{3.24}$$

where $\Phi_0(x) = 0$ and premium rate $c_0 > \mathrm{E}[Y_1] = 1/\lambda$. This result is the same as in Chan and Zhang (2006).

Definition 3.2 (Sub-adjustment coefficient)**.** Let $c_0 > 0$ and Y be a non-negative random variable. If there exists $h_0 > 0$ such that

$$\mathrm{E}[e^{h_0 Y}] \leq e^{h_0 c_0}, \tag{3.25}$$

then h_0 is called a *sub-adjustment coefficient of* (c_0, Y). Specifically, if (3.25) is an equality, then h_0 is called an *adjustment coefficient of* (c_0, Y).

Theorem 3.4. *Let $c_0 > 0$ be a premium rate and $\{Y_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process. If $h_0 > 0$ is a sub-adjustment coefficient of (c_0, Y_1), i.e.,*

$$\mathrm{E}[e^{h_0 Y_1}] \leq e^{h_0 c_0}, \tag{3.26}$$

then

$$\Phi_n(x) \leq e^{-h_0 x} \tag{3.27}$$

for all $x \geq 0$ and $n \in \mathbb{N}$.

Proof. Assume that $h_0 > 0$ is a sub-adjustment coefficient of (c_0, Y_1), i.e.,

$$\mathrm{E}[e^{h_0 Y_1}] \leq e^{h_0 c_0}.$$

We will prove this theorem by induction. We start with $n = 1$. By Chebyshev's inequality (C.8), we obtain

$$\Phi_1(x) = \mathrm{P}(Y_1 > x + c_0) = \mathrm{P}(e^{h_0 Y_1} > e^{h_0(x+c_0)}) \leq \frac{\mathrm{E}[e^{h_0 Y_1}]}{e^{h_0(x+c_0)}} \leq e^{-h_0 x}. \tag{3.28}$$

This proves (5.7) for $n = 1$. Assume that (5.7) holds for $n = k \geq 1$. By Lemma 3.3, we have

$$\Phi_{k+1}(x) = \Phi_1(x) + \int_{-\infty}^{x+c_0} \Phi_k(x + c_0 - y) dF_{Y_1}(y). \tag{3.29}$$

Firstly, we consider the second term of the right-hand side of (3.29). By using the inductive assumption, we have

$$\int_{-\infty}^{x+c_0} \Phi_k(x + c_0 - y) dF_{Y_1}(y) \leq \int_{-\infty}^{x+c_0} e^{-h_0(x+c_0-y)} dF_{Y_1}(y). \tag{3.30}$$

Now we consider the first term of the right-hand side of (3.29). By Chebyshev's inequality (Theorem C.8) again, we obtain

$$\begin{aligned} \mathrm{P}(Y_1 > x + c_0) &= \mathrm{P}(e^{h_0 Y_1}\mathbb{I}_{(x+c_0,\infty)}(Y_1) > e^{h_0(x+c_0)}) \\ &\le \frac{\mathrm{E}[e^{h_0 Y_1}\mathbb{I}_{(x+c_0,\infty)}(Y_1)]}{e^{h_0(x+c_0)}} \\ &= \int_{x+c_0}^{\infty} e^{-h_0(x+c_0-y)} dF_{Y_1}(y). \end{aligned} \tag{3.31}$$

Therefore, (3.29) becomes

$$\begin{aligned} \Phi_{k+1}(x) &\le \int_{-\infty}^{x+c_0} e^{-h_0(x+c_0-y)} dF_{Y_1}(y) + \int_{x+c_0}^{\infty} e^{-h_0(x+c_0-y)} dF_{Y_1}(y) \\ &= \int_{-\infty}^{\infty} e^{-h_0(x+c_0-y)} dF_{Y_1}(y) \\ &= \frac{e^{-h_0 x}}{e^{-h_0 c}} \int_{-\infty}^{\infty} e^{h_0 y} dF_{Y_1}(y) \\ &= e^{-h_0 x}\frac{\mathrm{E}[e^{h_0 Y_1}]}{e^{-h_0 c_0}} \\ &\le e^{-h_0 x}. \end{aligned} \tag{3.32}$$

This proves (5.7) for $n = k + 1$ and concludes the proof. □

Theorem 3.4 gives the following corollary:

Corollary 3.5. *Let $\alpha \in (0, 1)$ and $c_0 > 0$ be given and $\{Y_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process. If $h_0 > 0$ is a sub-adjustment coefficient of (c_0, Y_1), then*

$$u \ge -\frac{\log \alpha}{h_0}$$

is an acceptable initial capital corresponding $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ for all $N \in \mathbb{N}$.

Example 3.1 (Exponential claims). Consider the discrete-time surplus process (3.2) driven by the i.i.d. exponential claim size process $\{Y_n, n \in \mathbb{N}\}$ with intensity $\lambda > 0$, and the premium rate c_0, calculated by the expected value premium principle, i.e.,

$$c_0 = (1 + \theta_0)\mathrm{E}[Y_1] = (1 + \theta_0)/\lambda \tag{3.33}$$

where $\theta_0 > 0$ is the safety loading of the insurer. Next, we show that the adjustment coefficient of (c_0, Y_1) exists, i.e., there exists $h_0 > 0$ such that

$$\mathrm{E}[e^{h_0 Y_1}] = e^{h_0 c_0}.$$

Since

$$\lim_{h\to\lambda^-} \mathrm{E}[e^{hY_1}] = \lim_{h\to\lambda^-} \frac{\lambda}{\lambda - h} = \infty \tag{3.34}$$

and

$$\lim_{h\to\lambda^-} e^{hc_0} = \lim_{h\to\lambda^-} e^{h(1+\theta_0)/\lambda} = e^{1+\theta_0} < \infty, \tag{3.35}$$

there exists an $\varepsilon > 0$ such that $0 < \lambda - \varepsilon < \lambda$ and

$$\mathrm{E}[e^{h_1 Y_1}] > e^{h_1 c_0}$$

when $h_1 := \lambda - \varepsilon$. Let $\delta_n = \frac{\lambda}{n}, n \in \mathbb{N}$. Then,

$$\mathrm{E}[e^{\delta_n Y_1}] = \frac{\lambda}{\lambda - \delta_n} = \frac{n}{n-1} = 1 + \frac{1}{n-1}, n \geq 2. \tag{3.36}$$

By Taylor's expansion, we have

$$e^{\delta_n c_0} = e^{(1+\theta_0)/n} = 1 + \frac{1+\theta_0}{n} + \sum_{k=2}^{\infty} \frac{(1+\theta_0)^k}{n^k \cdot k!} \geq 1 + \frac{1+\theta_0}{n}. \tag{3.37}$$

Choosing $n_0 > \frac{1+\theta_0}{\theta_0} > 1$, i.e., $n_0\theta_0 - (1+\theta_0) > 0$, then

$$e^{\delta_{n_0} c_0} - \mathrm{E}[e^{\delta_{n_0} Y_1}] \geq \frac{1+\theta_0}{n_0} - \frac{1}{n_0 - 1} = \frac{n_0\theta_0 - (\theta_0 + 1)}{n_0(n_0 - 1)} > 0. \tag{3.38}$$

Since $\delta_n \to 0$ as $n \to \infty$, we can choose $\delta_{n_1} > 0$ such that $0 < \delta_{n_1} < h_1 < \lambda$ and $n_1 \geq n_0$. Let

$$f(h) = \mathrm{E}[e^{hY_1}] - e^{hc_0},$$

then $f(h)$ is continuous on $[\delta_{n_1}, h_1]$ and

$$f(\delta_{n_1}) > 0 \text{ and } f(h_1) < 0.$$

By the Bolzano's Theorem (Theorem E.6), there exists $h_0 \in (\delta_{n_1}, h_1)$ such that

$$f(h_0) = 0, \text{ i.e., } \mathrm{E}[e^{h_0 Y_1}] = e^{h_0 c}. \tag{3.39}$$

Thus, h_0 is an adjustment coefficient. By Theorem 3.4, we have

$$\Phi_N(x) \leq e^{-h_0 x} \tag{3.40}$$

for all $x \geq 0$ and $N \in \mathbb{N}$. Moreover, we get that h is a sub-adjustment coefficient of (c_0, Y_1) for all $0 < h \leq h_0$. This means that

$$\mathrm{E}[e^{hY_1}] \leq e^{hc_0} \tag{3.41}$$

for all $0 < h \leq h_0$. Next, we find the acceptable initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ in the case of $\alpha = 0.1$, $\lambda = 1$ and $c_0 = 1.1$. From the above arguments, there exists an adjustment coefficient r of $(1.1, Y_1)$ such that $0 < r < 1$, i.e.,

$$\frac{1}{1-r} = e^{1.1r}. \tag{3.42}$$

Now we approximate the adjustment coefficient r by a sub-adjustment coefficient. Since

$$0 < r - 0.176134 \leq 1/10^6, \tag{3.43}$$

then $0.176134 \in (0, r)$. Thus, we obtain that 0.176134 is a sub-adjustment coefficient of $(1.1, Y_1)$. By Corollary 3.5, we have that

$$x \geq -\frac{\log 0.1}{0.176134} = 13.072917, \tag{3.44}$$

which is an acceptable initial capital corresponding to $(0.1, N, 1.1, \{Y_n, n \in \mathbb{N}\})$ for all $N \in \mathbb{N}$. That is,

$$\Phi_N(x) \leq 0.1 \tag{3.45}$$

for all $x \geq 13.072917$ and $N \in \mathbb{N}$.

3.3 Existence of Minimum Initial Capital

A quantity α, discussed in previous section, can be described as the most acceptable probability that the insurance company will become insolvent. As a result of Corollary 3.2, we obtain that $\{x \geq 0 : \Phi_N(x) \leq \alpha\}$ is a non-empty set for all $N \in \mathbb{N}$. This means that we can always choose an initial capital which makes the value of ruin probability not exceed α. Since $\{x \geq 0 : \Phi_N(x) \leq \alpha\}$ is an infinite set, there are many acceptable initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$. In this section, we will prove the existence of

$$\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) = \min_{x \geq 0} \big\{x : \Phi_N(x) \leq \alpha\big\}. \tag{3.46}$$

Lemma 3.6. *Let a, b and α be real numbers such that $a \leq b$. If f is decreasing and right continuous on $[a, b]$ and $\alpha \in \big[f(b), f(a)\big]$, there exists $d \in [a, b]$ such that*

$$d = \min \big\{x \in [a, b] : f(x) \leq \alpha\big\}. \tag{3.47}$$

Proof. Let

$$S := \{x \in [a, b] : f(x) \leq \alpha\}. \tag{3.48}$$

Since $\alpha \in [f(b), f(a)]$, i.e., $f(b) \leq \alpha \leq f(a)$, we have $b \in S$. Thus, S is a non empty set. Since S is a subset of the closed and bounded interval $[a, b]$, there exists $d \in [a, b]$ such that $d = \inf S$. Next, we consider the following cases:

Case 1. $d = b$. We know that $b \in S$, thus $b = \min S$.

Case 2. $a \leq d < b$. Since $d = \inf S$, then there exists $d_n \in S$ such that

$$d \leq d_n < d + 1/n$$

for all $n \in \mathbb{N}$. Since f is decreasing and $d_n \in S$, we get

$$f(d_n) \leq \alpha.$$

Since f is right continuous at d, we have

$$f(d) = \lim_{n\to\infty} f(d_n) \le \alpha.$$

Therefore, $d \in S$, i.e., $d = \min S$. This completes the proof. $\square$

Theorem 3.7. *Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ and $c_0 > 0$. Then there exist $x^* \ge 0$ such that*

$$x^* = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}). \tag{3.49}$$

Proof. We consider by the following cases:

Case 1. $\Phi_N(0) \le \alpha$. We have $\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) = 0$.

Case 2. $\Phi_N(0) > \alpha$. By Lemma 3.2, there exists $\tilde{x} > 0$ such that

$$\Phi_N(\widetilde{x}) < \alpha, \text{ i.e., } \alpha \in \big[\Phi_N(\widetilde{x}), \Phi_N(0)\big].$$

Since $\Phi_N(x)$ is decreasing and right continuous, by Lemma 3.6, there exists $x^* \in [0, \tilde{x}]$ such that

$$x^* = \min_{x\in[0,\widetilde{x}]} \{x : \Phi_N(x) \le \alpha\} = \min_{x\in[0,\infty)} \{x : \Phi_N(x) \le \alpha\}.$$

That is, $x^* = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$. $\square$

Next, we will approximate the minimum initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ by applying the bisection technique for the decreasing and right continuous function.

Theorem 3.8. *Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ and $v_0, u_0 \ge 0$ such that $v_0 < u_0$. Let $\{u_n, n \in \mathbb{N}\}$ and $\{v_n, n \in \mathbb{N}\}$ be real sequences defined by*

$$\begin{cases} v_n = v_{n-1} & \text{and} \quad u_n = \frac{u_{n-1}+v_{n-1}}{2}, \quad \text{if } \Phi_N\left(\frac{u_{n-1}+v_{n-1}}{2}\right) \le \alpha, \\ v_n = \frac{v_{n-1}+u_{n-1}}{2} & \text{and} \quad u_n = u_{n-1}, \qquad\ \text{if } \Phi_N\left(\frac{u_{n-1}+v_{n-1}}{2}\right) > \alpha, \end{cases}$$

for all $n \in \mathbb{N}$. *If* $\Phi_N(u_0) \leq \alpha < \Phi_N(v_0)$, *then*

$$\lim_{n\to\infty} u_n = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) \tag{3.50}$$

and

$$0 \leq u_n - \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) \leq \frac{u_0 - v_0}{2^n}. \tag{3.51}$$

Proof. Obviously, $\{u_n, n \in \mathbb{N}\}$ is decreasing and $\{v_n, n \in \mathbb{N}\}$ is increasing. Moreover, $v_n \leq u_n$ for all $n \in \mathbb{N}$. Thus, $\{u_n, n \in \mathbb{N}\}$ and $\{v_n, n \in \mathbb{N}\}$ are convergent. Since

$$0 \leq u_n - v_n = \frac{(u_0 - v_0)}{2^n} \to 0 \text{ as } n \to \infty,$$

there exists $x^* \in [v_0, u_0]$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n := x^*. \tag{3.52}$$

Since $\Phi_N(x)$ is right continuous and $\Phi_N(u_n) \leq \alpha$ for all n, we have

$$\Phi_N(x^*) = \lim_{n\to\infty} \Phi_N(u_n) \leq \alpha. \tag{3.53}$$

Hence,

$$\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) \leq x^*. \tag{3.54}$$

Suppose that $\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) < x^*$. Then there exists $n_1 \in \mathbb{N}$ such that

$$\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) < v_n \leq x^*$$

for all $n \geq n_1$. Since $\Phi_N(x)$ is decreasing and $\Phi_N(v_n) > \alpha$ for all n, we have

$$\Phi_N(\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})) \geq \Phi_N(v_n) > \alpha.$$

But $\Phi_N(\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})) \leq \alpha$, which contradicts the definition of $\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$. Therefore, we conclude that

$$x^* = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}).$$

Since $v_n \le x^* \le u_n$, we have

$$0 \le u_n - x^* \le u_n - x^* + x^* - v_n = u_n - v_n = \frac{u_0 - v_0}{2^n} \tag{3.55}$$

for all $n \in \mathbb{N}_0$. This completes the proof. □

3.4 Numerical Results

We provide numerical illustrations of the main results. We approximate the minimum initial capital of the discrete-time surplus process (3.2) by using Theorem 3.8 in the case of $\{Y_n, n \in \mathbb{N}\}$, a sequence of i.i.d. exponential distribution with intensity $\lambda = 1$, by choosing model parameter combinations $\theta = 0.10$ and 0.25, i.e., $c_0 = 1.10$ and $c_0 = 1.25$, respectively; and $\alpha = 0.1$, 0.2, and 0.3.

Table 3.1 Minimum Initial Capital MIC($\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}$) in the Discrete-Time Surplus Process with Exponential Claims ($\lambda = 1$)

	$\alpha = 0.1$		$\alpha = 0.2$		$\alpha = 0.3$	
N	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$	$\theta = 0.10$	$\theta = 0.25$
10	4.31979	3.39733	2.89299	2.09365	1.99866	1.29822
20	5.80758	4.13270	3.98629	2.58739	2.84100	1.65475
30	6.79110	4.47565	4.69131	2.80480	3.37378	1.80598
40	7.52286	4.66050	5.20541	2.91736	3.75644	1.88242
50	8.09890	4.76750	5.60309	2.98062	4.04866	1.92467
100	9.81693	4.92645	6.74521	3.07094	4.86622	1.98378
200	11.13547	4.94953	7.56254	3.08341	5.42576	1.99174
300	11.60285	4.95022	7.83409	3.08377	5.60493	1.99197
400	11.79769	4.95025	7.94308	3.08379	5.67546	1.99198
500	11.88611	4.95025	7.99137	3.08379	5.70634	1.99198
1,000	11.96920	4.95025	8.03565	3.08379	5.73435	1.99198
5,000	11.97291	4.95025	8.03757	3.08379	5.73555	1.99198
10,000	11.97291	4.95025	8.03757	3.08379	5.73555	1.99198

Table 3.1 shows approximation of MIC($\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}$) with u_{25} as mentioned in Theorem 3.8 by choosing $v_0 = 0$ and $u_0 = 20$, and $\Phi_N(x)$ is computed from the recursive form (3.18).

Figure 3.1 Minimum Initial Capital MIC($\alpha, N, c, \{X_n, n \geq 1\}$) in the Discrete-Time Surplus Process with Exponential Claims ($\lambda = 1, N = 100$)

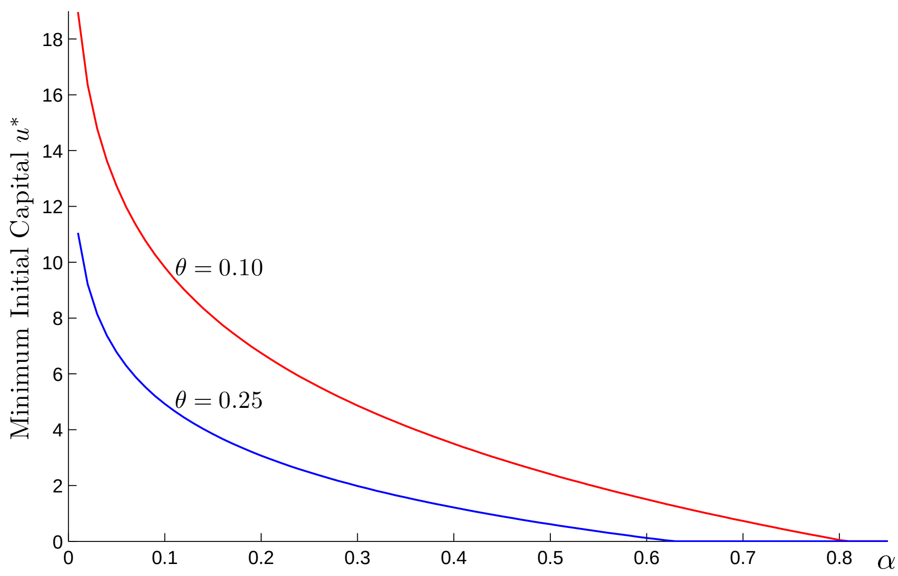

Figure 3.1 shows the approximation of MIC($\alpha, N, c, \{Y_n, n \geq 1\}$) for the various values of α with u_{25} as mentioned in Theorem 3.8. Here we choose $v_0 = 0$, $u_0 = 20$, and parameter combinations $\theta = 0.10$, $\theta = 0.25$, i.e., $c = 1.10$, $c = 1.25$, respectively.

CHAPTER IV

VALUE FUNCTION PROBLEM

In this chapter, we study a value function problem of a discrete-time surplus process under the investment and insurance controls. We derive the formula of the value function and prove the existence of an optimal plan.

4.1 Model Descriptions

In this section, we discuss the discrete-time surplus process under the conditions of reinsurance and investment. We assume that all of processes are defined in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$.

Firstly, we recall the discrete-time surplus process without control which has the claim size process $\{Y_n, n \in \mathbb{N}\}$ and claim arrival process $\{T_n, n \in \mathbb{N}_0\}$. Thus, we have the inter-arrival process $\{Z_n, n \in \mathbb{N}\}$ defined by

$$Z_n := T_n - T_{n-1} \tag{4.1}$$

which is the length of time between $(n-1)$th claim and nth claim. By period n, we mean the random interval $[T_{n-1}, T_n)\,, n \in \mathbb{N}$.

Now let the constant $c_0 > 0$ represent the premium rate for one unit time. The random variable

$$c_0 \sum_{i=1}^{n+1} Z_i = c_0 T_{n+1} \tag{4.2}$$

describes the inflow of capital into the business by time T_{n+1}, and $\sum_{i=1}^{n+1} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_{n+1}]$. Therefore, the quantity

$$\begin{aligned} X_{n+1} &= x + c_0 T_{n+1} - \sum_{i=1}^{n+1} Y_i \\ &= x + c_0 T_n - \sum_{i=1}^{n} Y_i + c_0(T_{n+1} - T_n) - Y_{n+1} \\ &= X_n + c_0 Z_{n+1} - Y_{n+1} \end{aligned} \tag{4.3}$$

is the surplus at time T_{n+1} and the constant $X_0 = x \geq 0$ is the initial capital.

In summary, the discrete time surplus process will be defined as follows:

$$X_0 = x,\ X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1},\ n \in \mathbb{N}. \tag{4.4}$$

Next, we discuss the discrete-time surplus process with reinsurance and investment controls. In the insurance business, reinsurance and investment are a normal activities of an insurance company because reinsurance can reduce the risk arising from claims, and an investment can increase the companies income. Thus, there are many papers studying their effect in the insurance business. For example, the effect of reinsurance on ruin probability was studied by Dickson and Waters (1996), minimizing the ruin probability in a continuous-time surplus process was considered by Browne (1995), Hipp and Plum (2000), Hipp and Vogt (2001), Højgaard and Taksar (1998a, 1998b), Schmidli (2001). We remark that a continuously controlled surplus process such as the Cramér-Lundberg model can be reduced to a discrete-time surplus process, for example, Schäl (2004).

In this chapter, we prove the existence of an optimal plan (the strategy or policy of choosing retention level of reinsurance and portfolio vector in investment for minimizing a value function) and derive a formula of the value function under the conditions that a reinsurer has the opportunity to default and investments in risky assets in the framework of a discrete-time surplus process.

Now, let $\{X_n, n \in \mathbb{N}_0\}$ be the surplus process which can be controlled by choosing the retention level b of reinsurance for one period, and at retention level

b, the insurer has to pay the premium rate to the reinsurer which is deducted from c_0, as a result of which the insurer's income rate will be represented by the function $c(b)$. The level $\bar{b}$ stands for the control action without reinsurance, so that $c_0 = c(\bar{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen. Of course, we obtain the *net income rate* $c(b)$ where

$$0 \leq c(\underline{b}) \leq c(b) \leq c(\bar{b}) = c_0$$

for all $b \in [\underline{b}, \bar{b}]$ and $c(b)$ is increasing. By the expected value premium principle, $c(b)$ can be calculated as follows:

$$c(b) = c_0 - (1 + \theta_1) \cdot \frac{\mathrm{E}[Y_1 - h(b, Y_1)]}{\mathrm{E}[Z_1]} \tag{4.5}$$

where $\theta_1 > 0$ is the safety loading of the reinsurer and the function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$, called *reinsurance recovery*, is paid by the reinsurer.

Next, we recall the *reinsurance credit risk* which is the risk of the reinsurance counterparty failing to pay reinsurance recoveries in full to the ceding company (insurer) in a timely manner, i.e., unwillingness to pay, or even not paying them at all. Therefore, we assume that for each retention level $b \in [\underline{b}, \bar{b}]$ the reinsurer has an opportunity to default, i.e., the insurer has to pay

$$\begin{cases} y & \text{if reinsurer default with probability } \mathrm{P}(K = 0) = p, \\ h(b, y) & \text{if reinsurer does not default with probability } \mathrm{P}(K = 1) = 1 - p, \end{cases}$$

where K is a random variable with value in $\{0, 1\}$ and $p \in [0, 1)$ is constant. The random variable K is said to be *binary recovery coefficient*. Let K_n be a binary recovery coefficient random variable at time $T_n, n \in \mathbb{N}$. Therefore, at time T_n the insurer pays

$$h(b_{n-1}, Y_n)K_n + Y_n(1 - K_n). \tag{4.6}$$

In addition, the insurer can invest the surplus (capital) in a financial market with m risky assets, called *stocks*, described by the price process

$$\left\{S_n = \left(S_n^1, S_n^2, ..., S_n^m\right), n \in \mathbb{N}_0\right\} \tag{4.7}$$

where $S_n^k > 0$ is the price of one share of stock k at the time T_n. We now define the return process

$$\left\{R_n = \left(R_n^1, R_n^2, ..., R_n^m\right), n \in \mathbb{N}\right\} \tag{4.8}$$

by

$$R_n^k = \frac{S_n^k - S_{n-1}^k}{S_{n-1}^k},$$

for all $k \in \{1, 2, 3, \cdots, m\}$. For each $n \in \mathbb{N}$, a portfolio vector

$$\delta_n = \left(\delta_n^1, \delta_n^2, \cdots, \delta_n^m\right) \in \mathbb{R}^m$$

specifies the time T_n and the component δ_n^k represents the amount invested in stock k during period $n+1$. This means that the insurance company holds δ_n^k/S_n^k shares of stock k during period $n+1$, so that the value of these share at the time T_{n+1} is

$$\frac{\delta_n^k}{S_n^k} \cdot S_{n+1}^k.$$

In this situation, we allow for a negative value for δ_n^k, that is, we admit the short selling of stocks. Let X_n be a surplus and (b_n, δ_n) be a control action at the time T_n. Then, we can modify the surplus process (4.4) as follows:

$$\begin{aligned}
X_{n+1} &= X_n + c(b_n)Z_{n+1} - \{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\} \\
&\quad - \sum_{k=1}^{m} \delta_n^k + \sum_{k=1}^{m} \frac{\delta_n^k}{S_n^k} S_{n+1}^k \\
&= X_n + c(b_n)Z_{n+1} - \{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\} \\
&\quad + \sum_{k=1}^{m} \delta_n^k \frac{(S_{n+1}^k - S_n^k)}{S_n^k} \\
&= X_n + c(b_n)Z_{n+1} - \{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\} + \sum_{k=1}^{m} \delta_n^k R_{n+1}^k \\
&= X_n + c(b_n)Z_{n+1} - \Big\{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\Big\} + \langle \delta_n, R_{n+1} \rangle
\end{aligned}$$

where $X_0 = x$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. It is convenient to set

$$X_0 = x, \;\; X_{n+1} = X_n + L(b_n, \delta_n, K_{n+1}, R_{n+1}, Y_{n+1}, Z_{n+1}), n \in \mathbb{N}_0 \tag{4.9}$$

where

$$L(b, \delta, k, r, y, z) = c(b) \cdot z - \big\{h(b, y)k + y(1 - k)\big\} + \langle \delta, r \rangle. \tag{4.10}$$

If we let

$$f(x, b, \delta, k, r, y, z) := x + L(b, \delta, r, y, z), \tag{4.11}$$

then f is the *system function* as mentioned in Bersekas and Shreve (1978). We see that the surplus process $\{X_n, n \in \mathbb{N}_0\}$ is driven by the sequence of control actions $\{(b_n, \delta_n), n \in \mathbb{N}_0\}$ and the sequence of random vectors $\{W_n, n \in \mathbb{N}\}$ where $W_n = (K_n, R_n, Y_n, Z_n)$ (the disturbance for period n) is the source of the randomness of the model. It is natural to assume that the process W_n is i.i.d., so we make the following assumption:

Assumption 1: Independence Assumption (IA)

$W_n = (K_n, R_n, Y_n, Z_n)$, $n \in \mathbb{N}$ are independent and identically distributed random variables (i.i.d.). In addition, it is assumed that (K_n, Y_n, Z_n) and R_n are independent for all $n \in \mathbb{N}$.

Let $k, l \in \mathbb{N}$ such that $k \neq l$ and A, B be Borel sets in $\mathbb{R}$. Under Assumption 1, we have

$$\begin{aligned}
&\mathrm{P}(R_k \in A, R_l \in B) \\
&= \mathrm{P}((K_k, R_k, Y_k, Z_k) \in \Omega \times A \times \Omega \times \Omega, (K_l, R_l, Y_l, Z_l) \in \Omega \times B \times \Omega \times \Omega) \\
&= \mathrm{P}((K_k, R_k, Y_k, Z_k) \in \Omega \times A \times \Omega \times \Omega)\mathrm{P}((K_l, R_l, Y_l, Z_l) \in \Omega \times B \times \Omega \times \Omega) \\
&= \mathrm{P}(R_k \in A)\mathrm{P}(R_l \in B)
\end{aligned}$$

and

$$\begin{aligned}\mathrm{P}(R_k \in A) &= \mathrm{P}((K_k, R_k, Y_k, Z_k) \in \Omega \times A \times \Omega \times \Omega) \\ &= \mathrm{P}((K_l, R_l, Y_l, Z_l) \in \Omega \times A \times \Omega \times \Omega) \\ &= \mathrm{P}(R_l \in A).\end{aligned}$$

This means that the process $\{R_n, n \in \mathbb{N}\}$ is i.i.d. Similarly, the processes $\{K_n, n \in \mathbb{N}\}$, $\{Y_n, n \in \mathbb{N}\}$ and $\{Z_n, n \in \mathbb{N}\}$ are also i.i.d.

4.2 Dynamic Programming with Finite Horizon

Let $\{X_n, n \in \mathbb{N}_0\}$ be a surplus process as mentioned in (4.9) with value in a state space $(S, \mathcal{S})$ which is a measurable space. Suppose that $\{X_n, n \in \mathbb{N}_0\}$ is driven by a sequence of i.i.d. random variables $\{W_n, n \in \mathbb{N}\}$ with values in a measurable space $(E, \mathcal{E})$. Here, $(E, \mathcal{E})$ is called the *disturbance space*. The surplus process can be controlled at the beginning of every period by a measurable space $(U, \mathcal{U})$ which is called the *control action space*. In addition, the model is specified by the following quantities:

- $\alpha \in [0, 1]$ is the *discount factor*;
- $g : S \times U \to (-\infty, \infty]$ is the *one-period cost function*, which is measurable and bounded from below;
- $N \in \mathbb{N}$ is a *time horizon* (number of periods) and
- $\widehat{V}_N : S \to (-\infty, \infty]$ is the *terminal cost function* for time horizon N, which is measurable and bounded from below.

Definition 4.1. A *plan* for the time horizon N over action space U is a (finite) sequence

$$\pi := (u_0, u_1, u_2, \cdots, u_{N-1}) = \{u_i\}_{i=0}^{N-1}$$

of control action $u_i \in U$ for all $i \in \{0, 1, 2, \cdots, N-1\}$. A set of all plans for the time horizon N over action space U is denoted by $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be *u-stationary*, if $\pi = (\underbrace{u, u, \cdots, u}_{N \text{ terms}})$ for some $u \in U$.

For each initial state $x \in S$ and plan $\pi = \{u_i\}_{i=0}^{N-1}$, the surplus process (4.9) can be written by

$$\begin{aligned} X_{n+1} &= X_n + L(u_n, W_{n+1}) \\ &= x + \sum_{k=0}^{n} L(u_k, W_{k+1}), n = 0, 1, 2, ..., N-1 \end{aligned} \tag{4.12}$$

and $X_0 = x$.

For the state $X_n = x_n$, the cost at the time T_n will be $g(x_n, u_n)$ and the next state

$$x_{n+1} = x_n + c(b_n)z_n - \{h(b_n, y_{n+1})k_{n+1} + y_{n+1}(1 - k_{n+1})\} + \langle \delta_n, r_{n+1} \rangle \tag{4.13}$$

will result in a cost $g(x_{n+1}, u_{n+1})$ at the time T_{n+1}. Thus, the present value of the costs at the time T_{n+1} will be $\alpha \cdot g(x_{n+1}, u_{n+1})$, i.e., $g(x_{n+1}, u_{n+1})$ is discounted by α.

Definition 4.2. Let N be the time horizon. Then the *total discounted cost function* and the *valued function for the time horizon N* are defined by

$$\Phi^{(N)}(x, \pi) = \mathrm{E}\left[\sum_{i=0}^{N-1} \alpha^i g\left(X_i, u_i\right) + \alpha^N \widehat{V}_N(X_N) \,|X_0 = x\right], \tag{4.14}$$

where $\pi = \{u_i\}_{i=0}^{N-1}$ and

$$V^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N,U)} \Phi^{(N)}(x, \pi), \quad \text{respectively.} \tag{4.15}$$

A plan $\pi \in \mathcal{P}(N, U)$ is said to be *optimal*, if

$$V^{(N)}(x) = \Phi^{(N)}(x, \pi). \tag{4.16}$$

If π is u-stationary, we write

$$\Phi^{(N)}(x, u) := \Phi^{(N)}(x, \pi). \tag{4.17}$$

4.3 Main Results

In this section, we study the insurance model introduced in Section 4.1 under the assumption that the insurer can borrow an unlimited amount of money. Let the state space $S = \mathbb{R}$ and the control space $U = [\underline{b}, \overline{b}] \times \mathbb{R}^m$. Thus, for each state $x \in S$, we can choose any control actions $u = (b, \delta) \in [\underline{b}, \overline{b}] \times \mathbb{R}^m$, where b is the retention level of reinsurance and $\delta = (\delta^1, \delta^2, \cdots, \delta^m)$ is the portfolio vector.

We study the cost structure which is given by the idea that the insurance company is not insolvent (ruined) but only penalized if the size of the surplus is negative or small. The penalty cost of being in state x is of the form const $\times\, e^{-\beta x}$ for some $\beta > 0$ (β is called a *cost level*). Therefore, we define the cost functions as

$$g(x, u) = \gamma \cdot e^{-\beta x}, \widehat{V}_N(x) = \nu_0 \cdot e^{-\beta x}, \text{ for some } \gamma, \nu_0 \geq 0, \tag{4.18}$$

when $x \in S, u \in U$. Thus, we obtain the total discounted cost function of model (4.9) as

$$\Phi^{(N)}(x, \pi) = \mathrm{E}\Big[\sum_{i=0}^{N-1} \alpha^i \gamma \cdot e^{-\beta X_i} + \alpha^N \nu_0 \cdot e^{-\beta X_N} \big| X_0 = x\Big], \tag{4.19}$$

where $\pi \in \mathcal{P}(N, U)$.

In this section, we will use the method of dynamic programming to prove the main theorem. In order to do this, we define $\Phi_n^{(N)}(x, \pi)$ and $V_n^{(N)}(x)$ as follows:

$$\begin{aligned}
\Phi_n^{(N)}(x, \pi) \quad &:= \quad \mathrm{E}\left[\sum_{i=n}^{N-1} \alpha^{i-n} \gamma \cdot e^{-\beta X_i} + \alpha^{N-n} \nu_0 \cdot e^{-\beta X_N} \,|X_n = x\right], && (4.20)\\
&\quad\ (n = 0, 1, 2, \cdots, N-1) \\
\Phi_N^{(N)}(x, \pi) \quad &:= \quad \nu_0 \cdot e^{-\beta x} && (4.21)
\end{aligned}$$

where $\pi = \{u_i\}_{i=0}^{N-1} \in \mathcal{P}(N, U)$ and

$$\begin{aligned}
V_n^{(N)}(x) \quad &= \quad \inf_{\pi \in \mathcal{P}(N,U)} \Phi_n^{(N)}(x, \pi), \ n = 0, 1, 2, \cdots, N-1, && (4.22)\\
V_N^{(N)}(x) \quad &= \quad \Phi_N^{(N)}(x, \pi). && (4.23)
\end{aligned}$$

It is obvious to see that

$$\Phi^{(N)}(x,\pi) = \Phi_0^{(N)}(x,\pi),\ \pi \in \mathcal{P}(N,U), \tag{4.24}$$

and

$$V^{(N)}(x) = V_0^{(N)}(x). \tag{4.25}$$

For each $\pi = (u_0, u_1, u_2, ...u_{N-1}) \in \mathcal{P}(N,U)$, we can see from equation (4.20) that

$$\begin{aligned} \Phi_n^{(N)}(x,\pi) &= \Phi_n^{(N)}(x,(u_0,u_1,u_2,...,u_{n-1},u_n,\cdots,u_{N-1})) \\ &= \mathrm{E}\left[\sum_{i=n}^{N-1} \alpha^{i-n}\gamma \cdot e^{-\beta X_i} + \alpha^{N-n}\nu_0 \cdot e^{-\beta X_N} | X_n = x\right] \end{aligned} \tag{4.26}$$

does not depend on the control actions $u_0, u_1, ..., u_{n-1}$. Therefore, (4.22) becomes

$$V_n^{(N)}(x) = \inf_{u_n,u_{n+1},\cdots,u_{N-1}\in U} \Phi_n^{(N)}(x,(u_0,u_1,u_2,...,u_{n-1},u_n,\cdots,u_{N-1})). \tag{4.27}$$

Next, we define a function $G : U \to [0,\infty]$ by

$$\begin{aligned} G(u) &:= \mathrm{E}\left[e^{-\beta L(u,W_1)}\right] \\ &= \mathrm{E}\left[e^{-\beta(c(b)Z_1 - h(b,Y_1)K_1 + Y_1(1-K_1) + <\delta,R_1>)}\right] \end{aligned} \tag{4.28}$$

for all $u = (b,\delta) \in U$ where W_1 is given as in Assumption 1 (IA). Thus, by Assumption 1 (IA), we have

$$\mathrm{E}\left[e^{-\beta L(u,W_n)}\right] = \mathrm{E}\left[e^{-\beta L(u,W_1)}\right] \tag{4.29}$$

for all $u \in U$ and $n \in \mathbb{N}$.

Remark 4.1. By Assumption 1 (IA), for each $\pi = \{u_i\}_{i=0}^{N-1} \in \mathcal{P}(N,U)$, the equation (4.20) becomes

$$\Phi_n^{(N)}(x,\pi) = \gamma e^{-\beta x} + \alpha G(u_n)\Phi_{n+1}^{(N)}(x,\pi) \tag{4.30}$$

for all $n \in \{0, 1, 2, \cdots, N-1\}$.

Proof of the Remark 4.1. Let $\pi = \{u_i\}_{i=0}^{N-1} \in \mathcal{P}(N, U)$.

In the case of $n = N-1$, we have

$$\begin{aligned}
\Phi_{N-1}^{(N)}(x,\pi) &= \mathrm{E}\Big[\gamma e^{-\beta x} + \alpha\nu_0 \cdot e^{-\beta(x+L(u_{N-1},W_N))}\Big] \\
&= \gamma e^{-\beta x} + \alpha \mathrm{E}\left[e^{-\beta L(u_{N-1},W_N)}\right]\nu_0 \cdot e^{-\beta x} \\
&= \gamma e^{-\beta x} + \alpha G(u_{N-1})\Phi_N^{(N)}(x,\pi). \qquad (4.31)
\end{aligned}$$

In the case of $0 \le n < N-1$. Consider

$$\begin{aligned}
&\Phi_n^{(N)}(x,\pi) \\
&= \mathrm{E}\Big[\sum_{i=n}^{N-1} \alpha^{i-n}\gamma e^{-\beta X_i} + \alpha^{N-n}\nu_0 \cdot e^{-\beta X_N} \,|X_n = x\Big] \\
&= \mathrm{E}\Big[\gamma e^{-\beta x} + \alpha\gamma e^{-\beta\left(x+L(u_n,W_{n+1})\right)} + \sum_{i=n+2}^{N-1} \alpha^{i-n}\gamma e^{-\beta\left(x+\sum_{j=n}^{i-1} L(u_j,W_{j+1})\right)} \\
&\qquad +\nu_0\alpha^{N-n} e^{-\beta\left(x+\sum_{j=n}^{N-1} L(u_j,W_{j+1})\right)}\Big] \\
&= \gamma e^{-\beta x} + \alpha \mathrm{E}\Big[e^{-\beta L(u_n,W_{n+1})}\Big\{\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma\alpha^{i-(n+1)} e^{-\beta\left(x+\sum_{j=n+1}^{i-1} L(u_j,W_{j+1})\right)} \\
&\qquad +\nu_0\alpha^{N-(n+1)} e^{-\beta\left(x+\sum_{j=n+1}^{N-1} L(u_j,W_{j+1})\right)}\Big\}\Big]. \qquad (4.32)
\end{aligned}$$

Since the $\{W_n, n \in \mathbb{N}\}$ is an independent sequence, $\{L(u_{n-1}, W_n), n \in \mathbb{N}\}$ is also an independent sequence. Thus, we obtain

$$\begin{aligned}
&\Phi_n^{(N)}(x,\pi) \\
&= \gamma e^{-\beta x} + \alpha \mathrm{E}\Big[e^{-\beta L(u_n,W_{n+1})}\Big]\mathrm{E}\Big[\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma\alpha^{i-(n+1)} e^{-\beta\left(x+\sum_{j=n+1}^{i-1} L(u_j,W_{j+1})\right)} \\
&\qquad +\nu_0\alpha^{N-(n+1)} e^{-\beta\left(x+\sum_{j=n+1}^{N-1} L(u_j,W_{j+1})\right)}\Big] \\
&= \gamma e^{-\beta x} + \alpha \mathrm{E}\Big[e^{-\beta L(u_n,W_{n+1})}\Big]\mathrm{E}\Big[\sum_{i=n+1}^{N-1} \alpha^{i-(n+1)}\gamma e^{-\beta X_i} + \alpha^{N-(n+1)}\nu_0 e^{-\beta X_N}\Big|X_{n+1} = x\Big] \\
&= \gamma e^{-\beta x} + \alpha G\left(u_n\right)\Phi_{n+1}^{(N)}(x,\pi). \qquad (4.33)
\end{aligned}$$

This proves Remark 4.1. □

Remark 4.1 leads to the following lemma:

Lemma 4.1. *Under Assumption 1, let $x \in S$ be an initial state and $u \in U$ be a control action. If $G(u) < \infty$, then*

$$\Phi_n^{(N)}(x,u) = \begin{cases} \left(\gamma - \left[\gamma - \nu_0(1-\alpha G(u))\right](\alpha G(u))^{N-n}\right) \cdot \frac{e^{-\beta x}}{1-\alpha G(u)}, & \alpha G(u) \neq 1, \\ \left(\gamma(N-n) + \nu_0\right) \cdot e^{-\beta x}, & \alpha G(u) = 1 \end{cases}$$

for all $n \in \{0, 1, 2, \cdots, N\}$.

Proof. In the case of $\alpha G(u) \neq 1$, we will prove this case by using mathematical induction. Obviously, the case $n = N$ holds. Now assume that

$$\Phi_n^{(N)}(x,u) = \left(\gamma - \left[\gamma - \nu_0 \cdot (1-\alpha G(u))\right](\alpha G(u))^{N-n}\right) \cdot \frac{e^{-\beta x}}{1-\alpha G(u)}, \tag{4.34}$$

holds for $n = k+1 \leq N$. By virtue of Remark 4.1, we get

$$\begin{aligned} &\Phi_k^{(N)}(x,u) \\ &= \gamma e^{-\beta x} + \alpha G(u)\Phi_{k+1}^{(N)}(x,u) \\ &= \gamma e^{-\beta x} + \alpha G(u)\left(\gamma - \left[\gamma - \nu_0(1-\alpha G(u))\right](\alpha G(u))^{N-(k+1)}\right) \cdot \frac{e^{-\beta x}}{1-\alpha G(u)} \\ &= \left(\gamma(1-\alpha G(u)) + \left(\gamma\alpha G(u) - \left[\gamma - \nu_0(1-\alpha G(u))\right](\alpha G(u))^{N-k}\right)\right) \cdot \frac{e^{-\beta x}}{1-\alpha G(u)} \\ &= \left(\gamma - \left[\gamma - \nu_0(1-\alpha G(u))\right](\alpha G(u))^{N-k}\right) \cdot \frac{e^{-\beta x}}{1-\alpha G(u)} \end{aligned} \tag{4.35}$$

which proves for $n = k$. Thus, the case $\alpha G(u) \neq 1$ holds. Similarly, the case $\alpha G(u) = 1$ also holds. This proves Lemma 4.1. □

Lemma 4.2. *Under Assumption 1, let $x \in S$ be an initial state. If there exists $u^* \in U$ such that*

$$G(u^*) = \min_{u \in U} \mathrm{E}\left[e^{-\beta L(u,W_1)}\right] < \infty, \tag{4.36}$$

then

$$V_n^{(N)}(x) = \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x) \tag{4.37}$$

and the u^*-stationary plan is an optimal plan, i.e.,

$$V^{(N)}(x) = \Phi^{(N)}(x, u^*).$$

Proof. Assume that there exists $u^* \in U$ satisfying the condition in Lemma 4.2. Let $n \in \{0, 1, 2, \cdots, N-1\}$. Then, by equation (4.27) and Remark 4.1, we have

$$\begin{aligned} V_n^{(N)}(x) &= \inf_{\pi \in \mathcal{P}(N,U)} \Phi_n^{(N)}(x, \pi) \\ &= \inf_{u_n, u_{n+1}, \ldots, u_{N-1} \in U} \Phi_n^{(N)}(x, (u_0, u_1, u_2, \cdots, u_{N-1})) \\ &= \gamma e^{-\beta x} + \alpha \inf_{u_n, u_{n+1}, \ldots, u_{N-1} \in U} \left\{ G(u_n) \Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \ldots u_{N-1})) \right\}. \end{aligned} \tag{4.38}$$

For each $\{u_i\}_{i=0}^{N-1} \in \mathcal{P}(N, U)$, we have $\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \cdots, u_{N-1})) \geq 0$ and $G(u_n) \geq 0$ for all $n \in \{0, 1, 2, \cdots, N-1\}$, and $\Phi_{n+1}^{(N-1)}(x, (u_0, u_1, u_2, \cdots, u_{N-1}))$ does not depend on the control actions $u_0, u_1, \cdots, u_n$. Therefore, (4.38) becomes

$$\begin{aligned} V_n^{(N)}(x) &= \gamma e^{-\beta x} + \alpha \inf_{u_n \in U} G(u_n) \cdot \inf_{u_{n+1}, \ldots, u_{N-1} \in U} \Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \ldots u_{N-1})) \\ &= \gamma e^{-\beta x} + \alpha G(u^*) \cdot \inf_{\pi \in \mathcal{P}(N,U)} \Phi_{n+1}^{(N)}(x, \pi) \\ &= \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x). \end{aligned} \tag{4.39}$$

Next, we will prove

$$V_n^{(N)}(x) = \Phi_n^{(N)}(x, u^*), 0 \leq n \leq N \tag{4.40}$$

by using mathematical induction. We start with $n = N$. By equation (4.23), assumption 4.40 follows. Now, we assume that (4.40) holds for $n = k + 1 \leq N$, i.e.,

$$V_{k+1}^{(N)}(x) = \Phi_{k+1}^{(N)}(x, u^*). \tag{4.41}$$

Then

$$\begin{aligned} V_k^{(N)}(x) &= \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{k+1}^{(N)}(x) \\ &= \gamma e^{-\beta x} + \alpha G(u^*) \cdot \Phi_{k+1}^{(N)}(x). \end{aligned} \tag{4.42}$$

From Remark 4.1, we obtain

$$\Phi_k^{(N)} = \gamma e^{-\beta x} + \alpha G(u^*) \cdot \Phi_{k+1}^{(N)}(x). \tag{4.43}$$

Thus,

$$V_k^{(N)}(x) = \Phi_k^{(N)}(x, u^*) \tag{4.44}$$

which proves (4.40) for $n = k$. We conclude that

$$V_n^{(N)}(x) = \Phi_n^{(N)}(x, u^*) \tag{4.45}$$

for all $n \in \{0, 1, 2, \cdots, N-1\}$. This implies that

$$V^{(N)}(x) = V_0^{(N)}(x) = \Phi_0^{(N)}(x, u^*) = \Phi^{(N)}(x, u^*), \tag{4.46}$$

i.e., u^*-stationary is an optimal plan. □

From Lemma 4.2, we need the condition for the existence of $\min_{u \in U} G(u)$ which can be shown by using the maximum and minimum theorem (Theorem E.4). Firstly, we need the property that $u \to G(u)$ is continuous, so we make the following assumption:

Assumption 2. Continuity Assumption (CA)

The functions $c(b)$ and $h(b, y)$ are continuous in b (for each y) and

$$\mathrm{E}\Big[e^{\beta \cdot Y_1}\Big] < \infty, \ \ \mathrm{E}\Big[e^{\varepsilon \cdot \|R_1\|}\Big] < \infty, \ \text{for all } \varepsilon > 0, \tag{4.47}$$

when $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^m$.

Lemma 4.3. *Under Assumption 1-2, the function $G : U \to [0, \infty]$, defined by (4.28), is continuous. Moreover,*

$$b \mapsto \mathrm{E}\big[e^{-\beta(c(b)Z_1 - \{h(b,Y_1)K_1 + Y_1(1-K_1)\})}\big] \ \textit{and} \ \delta \mapsto \mathrm{E}\big[e^{-\beta\langle \delta, R_1 \rangle}\big]$$

are also continuous.

Proof. Let $u = (b, \delta) \in U$ be given. Since $c(b)Z_1 \geq 0$ and

$$0 \leq h(b, Y_1)K_1 + Y_1(1 - K_1) \leq Y_1K_1 + Y_1(1 - K_1) = Y_1, \tag{4.48}$$

we have

$$c(b)Z_1 - \Big\{h(b, Y_1)K_1 + Y_1(1 - K_1)\Big\} + \langle \delta, R_1 \rangle \geq -Y_1 + \langle \delta, R_1 \rangle \tag{4.49}$$

and

$$\begin{aligned} G(u) &= \mathrm{E}\left[e^{-\beta(c(b)Z_1 - \left\{h(b,Y_1)K_1 + Y_1(1-K_1)\right\} + \langle \delta, R_1 \rangle)}\right] \\ &\leq \mathrm{E}\left[e^{-\beta(-Y_1 + \langle \delta, R_1 \rangle)}\right] \\ &= \mathrm{E}\left[e^{\beta Y_1 - \beta \langle \delta, R_1 \rangle}\right]. \end{aligned} \tag{4.50}$$

By Assumption 1 and 2, we have

$$G(u) \leq \mathrm{E}\left[e^{\beta Y_1}\right] \mathrm{E}\left[e^{-\beta \langle \delta, R_1 \rangle}\right] \leq \mathrm{E}\Big[e^{\beta Y_1}\Big] \cdot \mathrm{E}\Big[e^{|\beta| \cdot \|\delta\| \cdot \|R_1\|}\Big] < \infty. \tag{4.51}$$

Thus, by using the Lebesgue's dominated convergence theorem (Theorem C.2) and continuity of exponential function, we obtain

$$\begin{aligned} \lim_{v \to u} G(v) &= \lim_{(\tilde{b}, \tilde{\delta}) \to (b, \delta)} \mathrm{E}\left[e^{-\beta(c(\tilde{b})Z_1 - \left\{h(\tilde{b},Y_1)K_1 + Y_1(1-K_1)\right\} + \langle \tilde{\delta}, R_1 \rangle)}\right] \\ &= \mathrm{E}\left[\lim_{(\tilde{b}, \tilde{\delta}) \to (b, \delta)} e^{-\beta(c(\tilde{b})Z_1 - \left\{h(\tilde{b},Y_1)K_1 + Y_1(1-K_1)\right\} + \langle \tilde{\delta}, R_1 \rangle)}\right] \\ &= \mathrm{E}\left[e^{-\beta(c(b)Z_1 - \left\{h(b,Y_1)K_1 + Y_1(1-K_1)\right\} + \langle \delta, R_1 \rangle)}\right] \\ &= G(u). \end{aligned} \tag{4.52}$$

We now conclude that $u \mapsto G(u)$ is continuous. Similarly, we obtain that

$$b \mapsto \mathrm{E}\Big[e^{-\beta(c(b)Z_1 - \{h(b,Y_1)K_1 + Y_1(1-K_1)\})}\Big] \text{ and } \delta \mapsto \mathrm{E}\Big[e^{-\beta\langle \delta, R_1 \rangle}\Big]$$

are also continuous. $\square$

Assumption 3. No-Arbitrage Assumption (NA)

For any portfolio vector $\delta \in \mathbb{R}^m$, $\mathrm{P}\left(\langle \delta, R_1 \rangle \geq 0\right) = 1$ implies $\mathrm{P}\left(\langle \delta, R_1 \rangle = 0\right) = 1$.

In the investment, the investor will look for the *arbitrage opportunity*, i.e., they want to hold the portfolio $\delta_0 \in \mathbb{R}^m$ such that $\mathrm{P}\left(\langle \delta_0, R_1 \rangle \geq 0\right) = 1$, which implies that for the initial surplus $X_0 = x$, we have

$$\begin{aligned} X_1 &= x + c(b_0)Z_1 - \left\{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\right\} + \langle \delta_0, R_1 \rangle \\ &\geq x + c(b_0)Z_1 - \left\{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\right\} \text{ a.s.} \end{aligned} \tag{4.53}$$

which means that the portfolio $\delta_0 \in \mathbb{R}^m$ has no risk. Of course, the investor would like to use this opportunity because the quantity $\mathrm{P}\left(\langle \delta_0, R_1 \rangle > 0\right)$ may be positive which indicates an arbitrage opportunity. Note that Assumption 3 (NA) is equivalent to

"for any portfolio $\delta \in \mathbb{R}^m$, $0 < \mathrm{P}\left(\langle \delta, R_1 \rangle < 0\right) < 1$ or $\langle \delta, R_1 \rangle = 0$ a.s." (NA*)

By (NA*), we have

$$\mathbb{R}^m = \Im \cup \Im^* \text{ and } \Im \cap \Im^* \neq \emptyset$$

where

$$\Im = \left\{\delta \in \mathbb{R}^m : \langle \delta, R_1 \rangle = 0 \text{ a.s.}\right\}$$

and

$$\Im^* = \left\{\delta \in \mathbb{R}^m : 0 < \mathrm{P}\left(\langle \delta, R_1 \rangle < 0\right) < 1\right\}.$$

It is easy to see that $\Im$ is a linear subspace of $\mathbb{R}^m$. Thus, there exists a linear subspace $\Im^\perp$ of $\mathbb{R}^m$ such that

$$\mathbb{R}^m = \Im \oplus \Im^\perp \text{ and } \Im \cap \Im^\perp = \left\{0\right\}$$

($\mathbb{R}^m$ is the direct sum of $\Im$ and $\Im^\perp$) which implies $\Im^\perp \setminus \left\{0\right\} \subset \Im^*$.

Lemma 4.4. *Under Assumption 1-3, let $\delta \in \mathbb{R}^m$ be given. If $\delta \in \Im^{\perp} \setminus \{0\}$, then there exists an $\varepsilon > 0$ such that*

$$\mathrm{E}\Big[-\langle \delta, R_1 \rangle \, \mathbb{I}_{(\langle \delta, R_1 \rangle < 0)}\Big] \geq \varepsilon \cdot \mathrm{P}\Big(\langle \delta, R_1 \rangle \leq -\varepsilon\Big) > 0.$$

Proof. Let $\delta \in \Im^{\perp} \setminus \{0\}$. Then, by (NA*), we have

$$\mathrm{P}\left(\langle \delta, R_1 \rangle < 0\right) := q \tag{4.54}$$

for some $q > 0$. Let

$$A_n := \{\omega \in \Omega : \langle \delta, R_1(\omega) \rangle \leq -1/n\}$$

and

$$A_\infty := \{\omega \in \Omega : \langle \delta, R_1(\omega) \rangle < 0\}\,.$$

Obviously, $A_n \subset A_{n+1} \subset A_\infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = A_\infty$. Thus, $\{\mathrm{P}(A_n), n \in \mathbb{N}\}$ is an increasing sequence and

$$\lim_{l \to \infty} \mathrm{P}\left(A_l\right) = \lim_{l \to \infty} \mathrm{P}\Big(\bigcup_{n=1}^{l} A_n\Big) = \mathrm{P}\left(A_\infty\right) = q. \tag{4.55}$$

So that there exists $l_0 \in \mathbb{N}$ such that $\mathrm{P}(A_{l_0}) > q/2$, i.e.,

$$\mathrm{P}\big(\, \langle \delta, R_1 \rangle \leq -1/l_0\big) > q/2. \tag{4.56}$$

By Chebyshev's inequality (Theorem C.8), we have

$$\begin{aligned} l_0 \mathrm{E}\Big[-\langle \delta, R_1 \rangle \, 1_{(\langle \delta, R_1 \rangle < 0)}\Big] &\geq \mathrm{P}\big(-\langle \delta, R_1 \rangle \, 1_{(\langle \delta, R_1 \rangle < 0)} \geq 1/l_0\big) \\ &= \mathrm{P}\big(\langle \delta, R_1 \rangle \, 1_{(\langle \delta, R_1 \rangle < 0)} \leq -1/l_0\big) \\ &= \mathrm{P}\big(\langle \delta, R_1 \rangle \leq -1/l_0\big) \\ &> q_0/2 > 0. \end{aligned} \tag{4.57}$$

Choose $\varepsilon = 1/l_0$. The lemma follows. □

Theorem 4.5. *Under Assumption 1-3, let $x \geq 0$ be an initial capital. Then there exists $u^* = (b^*, \delta^*) \in U$ such that*

$$G(u^*) = \min_{(b,\delta)\in U} \mathrm{E}\Big[e^{-\beta\big(c(b)Z_1 - \big\{h(b,Y_1)K_1 + Y_1(1-K_1)\big\} + \langle \delta, R_1\rangle\big)}\Big] < \infty$$

and

$$V^{(N)}(x) = \begin{cases} \Big(\gamma - \left[\gamma - \nu_0 \cdot (1 - \alpha G(u^*))\right] (\alpha G(u^*))^N\Big) \cdot \frac{e^{-\beta x}}{1-\alpha G(u^*)}, & \alpha G(u^*) \neq 1, \\ (\gamma N + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

Moreover, u^*-stationary is an optimal plan.

Proof. By Assumption 1 (IA), we have

$$\inf_{u\in U} G(u) = \inf_{b\in\left[\underline{b},\overline{b}\right]} \mathrm{E}\Big[e^{-\beta\big(c(b)Z_1 - \big\{h(b,Y_1)K_1 + Y_1(1-K_1)\big\}\big)}\Big] \inf_{\delta\in\mathbb{R}^m} \mathrm{E}\Big[e^{-\beta\langle \delta, R_1\rangle}\Big].$$

Since $\left[\underline{b}, \overline{b}\right]$ is compact and

$$b \mapsto \mathrm{E}\Big[e^{-\beta\big(c(b)Z_1 - \big\{h(b,Y_1)K_1 + Y_1(1-K_1)\big\}\big)}\Big]$$

is continuous, by using the maximum and minimum theorem (Corollary E.4), there exists $b^* \in \left[\underline{b}, \overline{b}\right]$ such that

$$\mathrm{E}\Big[e^{-\beta\big(c(b^*)Z_1 - \big\{h(b^*,Y_1)K_1 + Y_1(1-K_1)\big\}\big)}\Big] = \min_{b\in[\underline{b},\overline{b}]} \mathrm{E}\Big[e^{-\beta\big(c(b)Z_1 - \big\{h(b,Y_1)K_1 + Y_1(1-K_1)\big\}\big)}\Big]. \tag{4.58}$$

Next, we will find the minimizer of $\mathrm{E}\big[e^{-\beta\langle\delta,R_1\rangle}\big]$ over $\mathbb{R}^m$. We consider the following cases:

Case1. $\Im = \mathbb{R}^m$. By (NA*), we can see that $\mathrm{E}\big[e^{-\beta\langle\delta,R_1\rangle}\big] = 1$ for all $\delta \in \mathbb{R}^m$.

Case2. $\Im \neq \mathbb{R}^m$. Then $\Im^\perp \neq \big\{0\big\}$. By Lemma 4.4, we can show that for each $\delta \in \Im^\perp \setminus \big\{0\big\}$, there exists an $\varepsilon > 0$ such that

$$\mathrm{E}\Big[-\langle \delta, R_1\rangle\, \mathbb{I}_{(\langle\delta,R_1\rangle<0)}\Big] \geq \varepsilon \cdot \mathrm{P}\Big(\langle \delta, R_1\rangle \leq -\varepsilon\Big) > 0.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{E}\Big[e^{-\beta\langle n\cdot\delta,R_1\rangle}\Big] &= \lim_{n\to\infty} \mathrm{E}\Big[e^{-\beta\langle n\cdot\delta,R_1\rangle}\mathbb{I}_{(\langle\delta,R_1\rangle<0)} + e^{-\beta\langle n\cdot\delta,R_1\rangle}\mathbb{I}_{(\langle\delta,R_1\rangle\geq 0)}\Big] \\
&\geq \lim_{n\to\infty} \mathrm{E}\left[e^{-\beta\langle n\cdot\delta,R_1\rangle}\mathbb{I}_{(\langle\delta,R_1\rangle<0)}\right] \\
&\geq \lim_{n\to\infty} e^{\beta n \mathrm{E}\left[-\langle\delta,R_1\rangle\mathbb{I}_{(\langle\delta,R_1\rangle<0)}\right]} \\
&\geq \lim_{n\to\infty} e^{\beta n\varepsilon \mathrm{P}(\langle\delta,R_1\rangle<-\varepsilon)} \\
&= \infty. \qquad (4.59)
\end{aligned}$$

Next, for each $\kappa > 0$, we define

$$F_\kappa := \left\{\delta \in \Im^\perp : \|\delta\| = 1,\ \mathrm{E}\left[e^{-\beta\langle\kappa\cdot\delta,R_1\rangle}\right] \leq 2\right\}.$$

Let κ_1 and κ_2 be two real numbers such that $\kappa_2 > \kappa_1 > 0$. If $F_{\kappa_2} \neq \emptyset$, by convexity of the exponential function, then

$$\begin{aligned}
\mathrm{E}\Big[e^{-\beta\langle\kappa_1\cdot\delta,R_1\rangle}\Big] &= \mathrm{E}\Big[e^{-\beta\frac{\kappa_1}{\kappa_2}\cdot\kappa_2\langle\delta,R_1\rangle + \frac{\kappa_2-\kappa_1}{\kappa_2}\cdot 0}\Big] \\
&\leq \frac{\kappa_1}{\kappa_2}\mathrm{E}\left[e^{-\beta\kappa_2\langle\delta,R_1\rangle}\right] + \frac{\kappa_2-\kappa_1}{\kappa_2} \\
&\leq \frac{2\kappa_1}{\kappa_2} + \frac{\kappa_2-\kappa_1}{\kappa_2} \\
&= \frac{\kappa_2+\kappa_1}{\kappa_2} \\
&< 2 \qquad (4.60)
\end{aligned}$$

for all $\delta \in F_{\kappa_2}$. This means that $F_{\kappa_1} \supset F_{\kappa_2}$ for all $\kappa_2 > \kappa_1 > 0$. By inequality (4.59), we have

$$\bigcap_{n\in\mathbb{N}} F_n = \emptyset.$$

Since F_κ is compact for all $\kappa > 0$, by Cantor's intersection theorem (Theorem E.5), there exists an $n_0 \in \mathbb{N}$ such that $F_n = \emptyset$ for all $n \geq n_0$. This implies that $F_\kappa = \emptyset$ for all $\kappa \geq n_0$ and this is equivalent to

$$\partial B_\kappa := \left\{\delta \in \Im^\perp : \|\delta\| = \kappa,\ \mathrm{E}\left[e^{-\beta\langle\delta,R_1\rangle}\right] \leq 2\right\} = \emptyset$$

for all $\kappa \geq n_0$. Therefore, we have

$$\begin{aligned}
\inf_{\delta \in \mathbb{R}^m} \mathrm{E}\left[e^{-\beta\langle\delta, R_1\rangle}\right] &= \inf_{\delta \in \mathbb{R}^m} \mathrm{E}\left[e^{-\beta(\langle\rho(\delta), R_1\rangle + \langle\delta - \rho(\delta), R_1\rangle)}\right] \\
&= \inf_{\delta \in \mathbb{R}^m} \mathrm{E}\left[e^{-\beta\langle\rho(\delta), R_1\rangle}\right] \\
&= \inf_{\delta \in \Im^{\perp}} \mathrm{E}\left[e^{-\beta\langle\delta, R_1\rangle}\right] \\
&= \inf_{\delta \in \Im^{\perp}, \|\delta\| \leq n_0} \mathrm{E}\left[e^{-\beta\langle\delta, R_1\rangle}\right], \qquad (4.61)
\end{aligned}$$

where $\rho : \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal projection on $\Im^{\perp}$. Since $\left\{\delta \in \Im^{\perp} : \|\delta\| \leq n_0\right\}$ is compact and $\delta \mapsto \mathrm{E}\left[e^{-\beta\langle\delta, R_1\rangle}\right]$ is continuous, there exists

$$\delta^* \in \left\{\delta \in \Im^{\perp} : \|\delta\| \leq n_0\right\}$$

such that

$$\mathrm{E}\left[e^{-\beta\langle\delta^*, R_1\rangle}\right] = \min_{\delta \in \Im^{\perp}, \|\delta\| \leq n_0} \mathrm{E}\left[e^{-\beta\langle\delta, R_1\rangle}\right]. \qquad (4.62)$$

Therefore, $u^* = (b^*, \delta^*)$ is a minimizer of $G(u)$. By Lemma 4.4, we see that u^*-stationary is an optimal plan. Also from Lemma 4.1, we obtain

$$V^{(N)}(x) = \begin{cases} \left(\gamma - \left[\gamma - \nu_0 \cdot (1 - \alpha G(u^*))\right] (\alpha G(u^*))^N\right) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \neq 1, \\ \left(\gamma N + \nu_0\right) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

This completes the proof. $\square$

Theorem 4.5 gives the following corollary.

Corollary 4.6. *Under Assumption 1-3, let $x \geq 0$ be an initial capital. If there exists $u^* \in U$ such that*

$$0 < \alpha G(u^*) < 1,$$

then

$$\lim_{N \to \infty} V^{(N)}(x) = \frac{\gamma e^{-\beta x}}{1 - \alpha G(u^*)}.$$

As a result of corollary 4.6, for large time horizon $N_0 \in \mathbb{N}$ and u^* a minimizer of $G(u)$ satisfying $0 < \alpha G(u^*) < 1$, the value function $V^{(N)}(x)$ can be approximated by $V^{(\infty)}(x)$ when

$$V^{(\infty)}(x) := \lim_{N\to\infty} V^{(N)}(x).$$

Next, we present an important special case of the value function which is defined by $g(x,u) = 0$, i.e., $\gamma = 0$, and $\nu_0 > 0$. This means that the insurance company has only to pay a penalty cost at the end. Thus, we want to minimize

$$\Phi^{(N)}(x,\pi) = \Phi_N^{(N)}(x,\pi) = \nu_0 \cdot e^{-\beta X_N},\ X_0 = x,$$

which is the same problem as maximizing the expected utility of terminal wealth if we choose the exponential utility function as $1 - e^{-\beta x}$.

Corollary 4.7. *Under Assumption 1-3, let $x \geq 0$ be an initial capital. If $g \equiv 0$, then there exists $u^* \in U$ such that*

$$V^{(N)}(x) = (\alpha G(u^*))^N \nu_0 \cdot e^{-\beta x}$$

where

$$G(u^*) = \min_{(b,\delta)\in U} \mathrm{E}\left[e^{-\beta\left(c(b)Z_1 - \left\{h(b,Y_1)K_1 + Y_1(1-K_1)\right\} + \langle\delta,R_1\rangle\right)}\right].$$

4.4 Simulation Results

In this section, we provide the simulation results of Theorem 4.5. We consider the discrete-time surplus process $\{X_n, n \in \mathbb{N}_0\}$ under the proportional reinsurer without investment, i.e.,

$$X_0 = x, X_n = X_{n-1} + c(b_{n-1})Z_n - \{b_{n-1} \cdot Y_n K_n + Y_n(1-K_n)\}$$

where $x \geq 0$ is an initial capital, the process $\{Y_n, n \in \mathbb{N}\}$ is i.i.d. exponential claim size with $\mathrm{E}[Y_1] = \lambda$, the process $\{Z_n, n \in \mathbb{N}\}$ is i.i.d. exponential inter-arrival with

$\mathrm{E}[Z_1] = \mu$, and the process $\{K_n, n \in \mathbb{N}\}$ is i.i.d. binary recovery coefficient with

$$\mathrm{P}(K_1 = 0) = p = 1 - \mathrm{P}(K_1 = 1).$$

The premium rate c_0 and $\tilde{c}(b)$ of insurer and reinsurer, respectively, are calculated by the expected value premium principle, i.e.,

$$c_0 = (1+\theta_0)\lambda/\mu \text{ and } \tilde{c}(b) = (1+\theta_1)(1-b)\lambda/\mu$$

where $\theta_0 > 0$ and $\theta_1 > 0$ a safety loading of insurer and reinsurer, respectively and b is the retention level. Now we fix $\theta_0 = 0.2$ and $\theta_1 = 0.3$, thus,

$$c(b) = c_0 - \tilde{c}(b) = (1.3b - 0.1)\lambda/\mu.$$

We set

$$\underline{b} = \frac{0.1\mu}{1.3\lambda} \text{ and } \bar{b} = 1.$$

In this situation, by Theorem 4.5, there exists b^* such that

$$G(b^*) = \min_{b \in [\underline{b}, \bar{b}]} \mathrm{E}\Big[e^{-\beta\left(c(b)Z_1 - \left\{h(b,Y_1)K_1 + Y_1(1-K_1)\right\}\right)}\Big] < \infty,$$

and b^*-stationary is an optimal plan. Specifically, b^* is called the *optimal retention level.*

Table 4.1 Optimal Retention Level

	$\mu = 0.5$		$\mu = 1.0$		$\mu = 2.0$	
β	$p = 0.1$	$p = 0$	$p = 0.1$	$p = 0$	$p = 0.1$	$p = 0$
0.20	0.951	0.708	0.839	0.598	0.817	0.590
0.40	0.528	0.373	0.471	0.225	0.460	0.314
0.60	0.403	0.262	0.362	0.178	0.364	0.222
0.80	0.367	0.206	0.333	0.150	0.362	0.176
	$\underline{b} = 0.154$		$\underline{b} = 0.077$		$\underline{b} = 0.039$	

Table 4.1 shows the approximation of the optimal retention level by choosing model parameter $\lambda = 1$ and parameter combinations $\beta = 0.2, 0.4, 0.6$ and 0.8, $\mu = 0.5, 1$ and 2, and $p = 0.0$ and 0.1.

Figure 4.1 Reinsurance without Default and Reinsurance Credit Risk

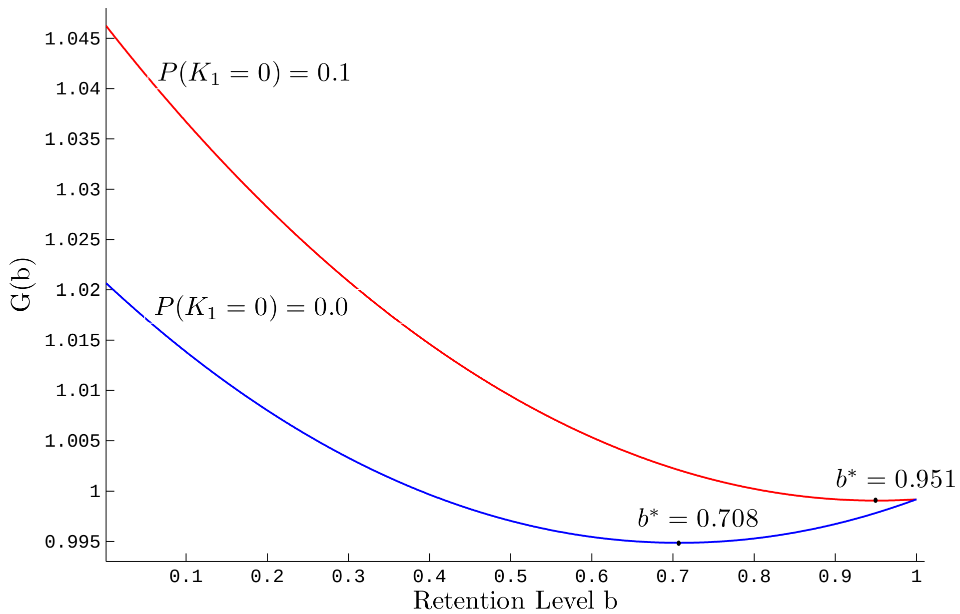

Figure 4.1 shows the approximation of $G(b)$ and the optimal retention level. We choose parameters $\beta = 0.2$, $\lambda = 1$ and $\mu = 0.5$, and parameter combinations $p = 0.0$ (reinsurance without default) and $p = 0.1$ (reinsurance credit risk).

Figure 4.2 Cost level and Optimal Retention Level

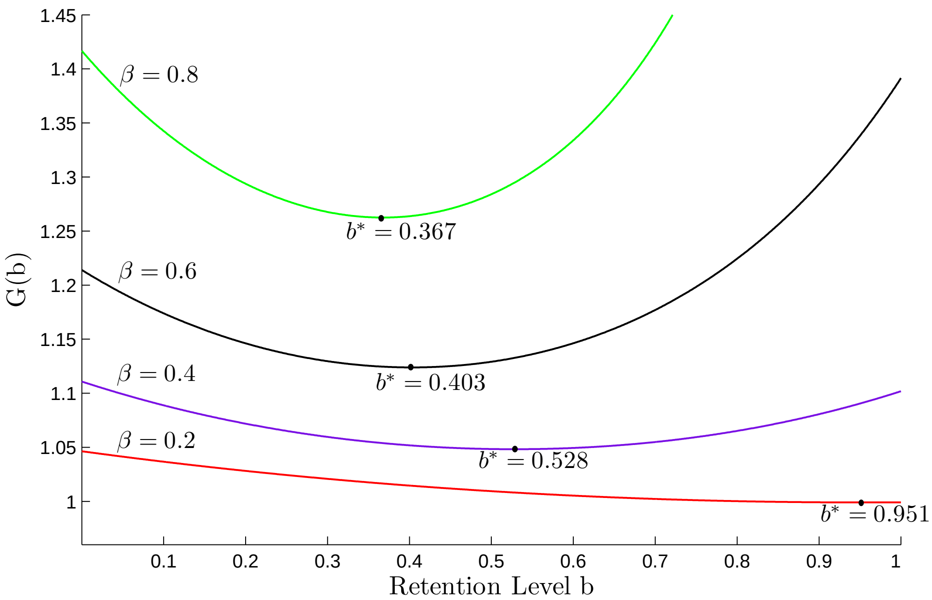

Figure 4.2 shows the approximation of $G(b)$ and the optimal retention level. We choose parameters $\lambda = 1$, $\mu = 0.5$ and $p = 0.1$, and parameter combinations $\beta = 0.2, 0.4, 0.6$ and 0.8.

CHAPTER V

CONCLUSIONS

This thesis is devoted to the study of the two different discrete-time surplus processes: one is considered the classical surplus process with the claim arrival times $T_n = n, n \in \mathbb{N}$ and the other is considered under the conditions of investment and reinsurance credit risk. Therefore, the results obtained are separated into two parts.

In the first part, the relationship between the initial capital and ruin probability of the discrete-time surplus process

$$X_0 = x, X_n = x + c_0 n + \sum_{i=1}^{n} Y_i, n \in \mathbb{N}, \tag{5.1}$$

where an initial capital $x \geq 0$ and the premium rate $c_0 > 0$, is studied. The claim size process $\{Y_n, n \in \mathbb{N}\}$ is assumed to be i.i.d. The ruin probability at one of the times $1, 2, 3, \cdots, N$ is defined by

$$\Phi_N(x) = \mathrm{P}\{X_i < 0 \text{ for some } i \in \{1, 2, 3, \cdots, N\} | X_0 = x\} \tag{5.2}$$

where $x \geq 0$ is an initial capital. $\{x \geq 0 : \Phi_N(x) \leq \alpha\}$ is the set of acceptable initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ and the minimum initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ is defined by

$$\mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) = \min_{x \geq 0}\{x : \Phi_N(x) \geq \alpha\}. \tag{5.3}$$

The main results of this part are summarized as follows:

Theorem 5.1. *Let $N \in \mathbb{N}$ and $c_0 > 0$ be given. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then*

$$\lim_{x \to \infty} \Phi_N(x) = 0. \tag{5.4}$$

Corollary 5.2. *Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ and $c_0 > 0$ be given. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then there exists $\tilde{x} \geq 0$ such that, for all $x \geq \tilde{x}$, x is an acceptable initial capital corresponding to $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$.*

Lemma 5.3. *Let $N \in \mathbb{N}$, $c_0 > 0$ and $x \geq 0$ be given. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then the ruin probability at one of the times $1, 2, 3, \cdots, N$ satisfies the following equation*

$$\Phi_N(x) = \Phi_1(x) + \int_{-\infty}^{x+c_0} \Phi_{N-1}(x + c_0 - y) dF_{Y_1}(y) \tag{5.5}$$

where $\Phi_0(x) = 0$.

Theorem 5.4. *Let $c_0 > 0$ be a premium rate and $\{Y_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process. If $h_0 > 0$ is a sub-adjustment coefficient of (c_0, Y_1), i.e.,*

$$\mathrm{E}[e^{h_0 Y_1}] \leq e^{h_0 c_0}, \tag{5.6}$$

then

$$\Phi_n(x) \leq e^{-h_0 x} \tag{5.7}$$

for all $x \geq 0$ and $n \in \mathbb{N}$.

Corollary 5.5. *Let $\alpha \in (0,1)$ and $c_0 > 0$ be given and $\{Y_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process. If $h_0 > 0$ is a sub-adjustment coefficient of (c_0, Y_1), then*

$$u \geq -\frac{\log \alpha}{h_0}$$

is an acceptable initial capital corresponding $(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\})$ for all $N \in \mathbb{N}$.

Theorem 5.6. *Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ and $c_0 > 0$. If $\{Y_n, n \in \mathbb{N}\}$ is an i.i.d. claim size process, then there exist $x^* \geq 0$ such that*

$$x^* = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}).$$

Theorem 5.7. *Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ $v_0, u_0 \geq 0$ such that $v_0 < u_0$. Let $\{Y_n, n \in \mathbb{N}\}$ be an i.i.d. claim size process, $\{u_n, n \in \mathbb{N}\}$ and $\{v_n, n \in \mathbb{N}\}$ be real sequences defined by*

$$\begin{cases} v_n = v_{n-1} & \text{and} \quad u_n = \frac{u_{n-1}+v_{n-1}}{2}, \quad \text{if } \Phi_N\left(\frac{u_{n-1}+v_{n-1}}{2}\right) \leq \alpha, \\ v_n = \frac{v_{n-1}+u_{n-1}}{2} & \text{and} \quad u_n = u_{n-1}, \qquad \text{if } \Phi_N\left(\frac{u_{n-1}+v_{n-1}}{2}\right) > \alpha, \end{cases}$$

for all $n \in \mathbb{N}$. If $\Phi_N(u_0) \leq \alpha < \Phi_N(v_0)$, then

$$\lim_{n\to\infty} u_n = \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) \tag{5.8}$$

and

$$0 \leq u_n - \mathrm{MIC}(\alpha, N, c_0, \{Y_n, n \in \mathbb{N}\}) \leq \frac{u_0 - v_0}{2^n}. \tag{5.9}$$

In the second part, we consider the discrete-time surplus process

$$X_{n+1} = X_n + c(b_n)Z_{n+1} - \left\{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\right\} + \langle \delta_n, R_{n+1}\rangle$$

for all $n \in \mathbb{N}$ and $X_0 = x \geq 0$ as initial capital. The surplus process $\{X_n, n \in \mathbb{N}\}$ is driven by the sequence of control actions $\{u_n = (b_n, \delta_n), n \in \mathbb{N}_0\}$ and the sequence of random vectors $\{W_n = (K_n, R_n, Y_n, Z_n), n \in \mathbb{N}\}$ where $\{K_n, n \in \mathbb{N}\}$ is the binary recovery coefficient process, $\{R_n, n \in \mathbb{N}\}$ is the return process, $\{Y_n, n \in \mathbb{N}\}$ is the claim size process, and $\{Z_n, n \in \mathbb{N}\}$ is the inter-arrival process. The total discounted cost function and the valued function for the time horizon $N \in \mathbb{N}$ are defined by

$$\Phi^{(N)}(x, \pi) = \mathrm{E}\left[\sum_{i=0}^{N-1} \alpha^i e^{-\beta X_i} + \alpha^N \nu_0 e^{-\beta X_N} \,|X_0 = x\right], \tag{5.10}$$

where $\pi = \{u_i, n \in \mathbb{N}\}$ and

$$V^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N,U)} \Phi^{(N)}(x, \pi), \quad \text{respectively}, \tag{5.11}$$

the control space $U = [\underline{b}, \overline{b}] \times \mathbb{R}^m$. They are considered under the following assumptions:

Assumption 1: Independence Assumption (IA)

$W_n = (K_n, R_n, Y_n, Z_n),\ n \in \mathbb{N}$ are independent and identically distributed random variables (i.i.d.). In addition, it is assumed that (K_n, Y_n, Z_n) and R_n are independent for all $n \in \mathbb{N}$.

Assumption 2. Continuity Assumption (CA)

The functions $c(b)$ and $h(b, y)$ are continuous in b (for each y) and

$$\mathrm{E}\Big[e^{\beta \cdot Y}\Big] < \infty,\ \ \mathrm{E}\Big[e^{\varepsilon \cdot \|R\|}\Big] < \infty,\ \text{for all } \varepsilon > 0.$$

Assumption 3. No-Arbitrage Assumption (NA)

For any portfolio vector $\delta \in \mathbb{R}^m$, $P\left(\langle \delta, R_1 \rangle \geq 0\right) = 1$ implies $P\left(\langle \delta, R_1 \rangle = 0\right) = 1$.

We obtain the following main theorem:

Theorem 5.8. *Under Assumption 1-3, let $x \geq 0$ be an initial capital. Then there exists $u^* = (b^*, \delta^*) \in U$ such that*

$$G(u^*) = \min_{(b,\delta)\in U} \mathrm{E}\Big[e^{-\beta\left(c(b)Z - \left\{h(b,Y)K + Y(1-K)\right\} + \langle \delta, R \rangle\right)}\Big] < \infty$$

and

$$V^{(N)}(x) = \begin{cases} \left(\gamma - \left[\gamma - \nu_0 \cdot (1 - \alpha G(u^*))\right] (\alpha G(u^*))^N\right) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \neq 1, \\ (\gamma N + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

Moreover, u^*-stationary is an optimal plan.

Finally, we should observe that further problems can be considered. For instance, the ruin probability problem under the reinsurance credit risk, the minimum initial problem in the situation that claim arrival times are random variables, the surplus control problem in order to reach the some target, etc. Furthermore, we can consider the total capital requirement (TCR) problem as the minimum initial capital problem. We will continue to study in this field.

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APPENDICES

APPENDIX A

Notations

$(\Omega, \mathcal{F}, \mathrm{P})$	Probability space
Ω	Outcome space
$\mathcal{F}$	σ-field
P	Probability measure
$\sigma(X)$	σ-field generated by random variable X
$(X \in B)$	$\{\omega \in \Omega : X(\omega) \in B\}$
$(X \leq x)$	$\{\omega \in \Omega : X(\omega) \leq x\}$
$\mathrm{E}[X]$	Expectation of the random variable X
$\mathbb{E}[X\|\mathcal{F}]$	Conditional expectation of the random variable X given the σ-field $\mathcal{F}$
$\mathbb{I}_A$	Indicator of set A
$\mathbb{N}$	Set of positive integers
$\mathbb{N}_0$	Set of non-negative integers
$\mathbb{R}$	Real line
$\mathbb{R}^m$	n-dimensional Euclidean space
inf	Infimum (greatest lower bound)
sup	Supremum (least upper bound)
$\\|x\\|$	Norm of x
$\langle x, y \rangle$	Inner product of x and y
$\Im^{\perp}$	Orthogonal complement of a closed subspace $\Im$
$x \mapsto f(x)$	x maps to $f(x)$

APPENDIX B

Computer Programs

This appendix contains a copy of the programs written in **Matlab** to implement the approximation in Chapter III and Chapter IV.

```
%----------- Table 3.1 in Chapter III -----------%

clc;clear;

%The number of iterations

n=25;

%Set the Parameters

lambda=1; alpha=0.1; c0=1.1;

fprintf('lambda=%7.1f,alpha=%5.5f,premuim=%1.2f\n',lambda,alpha,c0)

fprintf('-----------------------------------------\n')

%The number of claims

N=[10 20 30 40 50 100 200 300 400 500 1000 5000 10000];

for j=1:13

%Bisection method

v0=0;u0=20; u=[];v=[];w=[];

u(1)=u0;v(1)=v0;w(1)=u(1);

  for i=1:n

    w(i+1)=(u(i)+v(i))*0.5;

    A=[]; PH1=[];

    for k=1:N A(k)=log(k);

      PH1(k)=exp((k-1)*log(lambda*(w(i+1)+k*c0))+log(w(i+1)+c0)...
```

```
      -lambda*(w(i+1)+k*c0)-log(w(i+1)+k*c0)-(sum(A)-A(k)));
    end
    phi1=sum(PH1);
    if (phi1<=alpha)
      u(i+1)=w(i+1);
      v(i+1)=v(i);
    else
      v(i+1)=w(i+1);
      u(i+1)=u(i);
    end
%Output
M(j)=ceil(u(n)*100000)/100000;
fprintf('N=%7.1f, Minimum Initial Captial=%5.5f \n',N(j),M(j))
end
%----------- End -----------%
```

```
%----------- Figure 3.1 in Chapter III -----------%
clc;clear;
n=25;
N=100;
alpha=[];
v0=0;u0=20;
lambda=1;
c0=[1.1 1.25];
MIC=[];
for r=1:2
  for q=1:90
    alpha(q)=q/100;
    u=[];v=[];w=[];
    u(1)=u0;v(1)=v0;w(1)=u(1);
    for i=1:n
      w(i+1)=(u(i)+v(i))*0.5;
      A=[]; PH1=[];
      for k=1:N A(k)=log(k);
        PH1(k)=exp((k-1)*log(lambda*(w(i+1)+k*c0(r)))...
        +log(w(i+1)+c0(r))-lambda*(w(i+1)+k*c0(r))...
        -log(w(i+1)+k*c0(r))-(sum(A)-A(k)));
      end
      phi1=sum(PH1);
      if (phi1<=alpha(q))
        u(i+1)=w(i+1);
        v(i+1)=v(i);
```

```
      else
        v(i+1)=w(i+1);
        u(i+1)=u(i);
      end
    end
    MIC(r,q)=u(n)
  end
end
hold on;
t=0.01:0.01:0.9
plot(t,MIC(1,:),'r')
plot(t,MIC(2,:),'b')
%----------- End -----------%
```

```
%---- Table 4.1, Figure 4.1 and Figure 4.2 in Chapter IV ----%
clc;clear;
%Set the size of random numbers
N=10000;
%Set the means of claim size and inter-arrival time
lambda=1; mu=2.0;
%Generate the random numbers
Y=exprnd(lambda,1,N); Z=exprnd(mu,1,N);
K=binornd(1,0.9,1,N);
%set the parameters (beta)
beta1=0.2; beta2=0.4;beta3=0.6;beta4=0.8;
%Compute the premium rate of insurer with safety loading 0.2
c0=1.2*lambda/mu;
A1=[];A2=[];A3=[];A4=[];A5=[];A6=[];A7=[];A8=[];
b1=0;
for j=1:(1-b1)*1000+1
  b=b1;
  b=b+(j-1)/1000;
  %(1-b)*1.3*lambda/mu is premium rate of reinsurer
  %with safety loading 0.3
  cb=c0-(1-b)*1.3*lambda/mu;
  E1=[];E2=[];E3=[];E4=[];E5=[];E6=[];E7=[];E8=[];
    for i=1:N
      E1(i)=exp(-beta1*(cb*Z(i)-(b*Y(i)*K(i)+Y(i)*(1-K(i)))));
      E2(i)=exp(-beta1*(cb*Z(i)-b*Y(i)));
      E3(i)=exp(-beta2*(cb*Z(i)-(b*Y(i)*K(i)+Y(i)*(1-K(i)))));
```

```
      E4(i)=exp(-beta2*(cb*Z(i)-b*Y(i)));
      E5(i)=exp(-beta3*(cb*Z(i)-(b*Y(i)*K(i)+Y(i)*(1-K(i)))));
      E6(i)=exp(-beta3*(cb*Z(i)-b*Y(i)));
      E7(i)=exp(-beta4*(cb*Z(i)-(b*Y(i)*K(i)+Y(i)*(1-K(i)))));
      E8(i)=exp(-beta4*(cb*Z(i)-b*Y(i)));
    end
    A1(j)=mean(E1);
    A2(j)=mean(E2);
    A3(j)=mean(E3);
    A4(j)=mean(E4);
    A5(j)=mean(E5);
    A6(j)=mean(E6);
    A7(j)=mean(E7);
    A8(j)=mean(E8);
end
for j=1:(1-b1)*1000+1
  if A1(j)==min(A1)
    M1=(j-1)/1000;
  end
  if A2(j)==min(A2)
    M2=(j-1)/1000;
  end
  if A3(j)==min(A3)
    M3=(j-1)/1000;
  end
  if A4(j)==min(A4)
```

```
    M4=(j-1)/1000;
  end
  if A5(j)==min(A5)
    M5=(j-1)/1000;
  end
  if A6(j)==min(A6)
    M6=(j-1)/1000;
  end
  if A7(j)==min(A7)
    M7=(j-1)/1000;
  end
  if A8(j)==min(A8)
    M8=(j-1)/1000;
  end
end

%-Output for Table 4.1
fprintf('-----------------------------\n')
fprintf('   lambda=%1.1f,mu=%1.1f  \n',lambda,mu)
fprintf('-----------------------------\n')
fprintf('  beta |  p=0.1  | p=0.0\n')
fprintf('-----------------------------\n')
fprintf('  0.2  |  %1.3f  |  %1.3f   \n',M1,M2)
fprintf('  0.4  |  %1.3f  |  %1.3f   \n',M3,M4)
fprintf('  0.6  |  %1.3f  |  %1.3f   \n',M5,M6)
fprintf('  0.8  |  %1.3f  |  %1.3f   \n',M7,M8)
```

```
fprintf('-----------------------------\n')

%-Output for Figure 4.1
hold on;
t=b1:0.001:1;
%Line A: beta=0.2,lambda=1,mu=2,c0=0.6,p=0.1
plot(t,A1,'r')
%Line B: beta=0.4,lambda=1,mu=2,c0=0.6,p=0.1
plot(t,A3,'y')
%Line C: beta=0.6,lambda=1,mu=2,c0=0.6,p=0.1
plot(t,A5,'k')
%Line D: beta=0.8,lambda=1,mu=2,c0=0.6,p=0.1
plot(t,A7,'c')

%-Output for Figure 4.2
hold on;
t=b1:0.001:1;
%Line 1: beta=0.2,lambda=1,mu=2,c0=0.6,p=0.1
plot(t,A1,'r')
%Line 2: beta=0.2,lambda=1,mu=2,c0=0.6,p=0.0
plot(t,A2,'m')
%----------- End-----------%
```

APPENDIX C

Probability Theory

We recall some definition and theorem in probability theory. Most of these results can be found in Brzeźniak and Zastawniak (1999), Capiński and Kopp (2004), and Aggoun and Elliott (2004).

Definition C.1. Let Ω be a non-empty set. A *σ-field* $\mathcal{F}$ on Ω is a family of a subsets of Ω such that

1. the empty set $\emptyset$ belong to $\mathcal{F}$;
2. if A belong to $\mathcal{F}$, then so does the complement $\Omega \setminus A$;
3. if $A_1, A_2, \cdots$ is a sequence of sets in $\mathcal{F}$, then their union $A_1 \cup A_2 \cup \cdots$ also belong to $\mathcal{F}$.

Definition C.2. Let $\mathcal{F}$ be a σ-field on Ω. A *probability measure* P is a function

$$\mathrm{P} : \mathcal{F} \to [0,1]$$

such that

1. $\mathrm{P}(\Omega) = 1$;
2. if $A_1, A_2, \cdots$ are pairwise disjoint set (that is, $A_i \cap A_j = \emptyset$ for $i \neq j$) belong to $\mathcal{F}$, then

$$\mathrm{P}(A_1 \cup A_2 \cup \cdots) = \mathrm{P}(A_1) + \mathrm{P}(A_2) + \cdots .$$

The triple $(\Omega, \mathcal{F}, \mathrm{P})$ is called a *probability space*. The sets belonging to $\mathcal{F}$ is called *events*. An event A is said to occur *almost surely* (a.s.) whenever $\mathrm{P}(A) = 1$.

Definition C.3. If $\mathcal{F}$ is a σ-field on Ω, then a function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}$-*measurable* if

$$(X \in B) := \{\omega \in \Omega : X(\omega) \in B\} = X^{-1}(B) \in \mathcal{F}$$

for every Borel set $B \in \mathcal{B}(\mathbb{R})$. If $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, then such a function X is called a *random variable.*

Definition C.4. The σ-field $\sigma(X)$ *generated* by a random variable $X : \Omega \to \mathbb{R}$ consists of all sets of the form $(X \in B)$, where B is a Borel set in $\mathbb{R}$.

Lemma C.1 (Doob-Dynkin)**.** Let X be a random variable. Then each $\sigma(X)$-measurable random variable Y can be written as

$$Y = f(X)$$

for some Borel function $f : \mathbb{R} \to \mathbb{R}$.

Definition C.5. Every random variable $X : \Omega \to \mathbb{R}$ gives rise to a probability measure

$$\mathrm{P}_X(B) = \mathrm{P}(X \in B)$$

on $\mathbb{R}$ defined on the σ-field of Berel sets $B \in \mathcal{B}(\mathbb{R})$. We call P_X the distribution of X. The function $F_X : \mathbb{R} \to [0, 1]$ defined by

$$F_X(x) = \mathrm{P}(X \leq x)$$

is called the *distribution function* of X.

Definition C.6. If there is a Borel function $f_X : \mathbb{R} \to \mathbb{R}$ such that for any Borel set $B \subset \mathbb{R}$

$$\mathrm{P}(X \in B) = \int_B f_X(x)dx$$

then X is said to be a random variable with *absolutely continuous distribution* and f_X is called *density* of X. If there is a (finite or infinite) sequence of pairwise distinct real numbers $x_1, x_2, \cdots$ such that for any Borel set $B \subset \mathbb{R}$

$$\mathrm{P}(X \in B) = \sum_{x_i \in B} \mathrm{P}(X = x_i),$$

then X is said to have *discrete distribution* with value $x_1, x_2, \cdots$ and *mass* $\mathrm{P}(X = x_i)$ at x_i.

Definition C.7. A random variable $X : \Omega \to \mathbb{R}$ is said to be *integrable* if

$$\int_\Omega |X| d\mathrm{P} < \infty.$$

Then

$$\mathrm{E}[X] := \int_\Omega X d\mathrm{P}$$

exist and is called the *expectation* of X.

Definition C.8. Two events $A, B \in \mathcal{F}$ are called *independent* if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B).$$

In general, we say that n events $A_1, A_2, \cdots, A_n \in \mathcal{F}$ are *independent* if

$$\mathrm{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mathrm{P}(A_{i_1})\mathrm{P}(A_{i_2}) \cdots \mathrm{P}(A_{i_k})$$

for any indices $1 \leq i_1 < i_2 < \cdots < t_k \leq n$.

Definition C.9. Two random variable X and Y are called *independent* if for any Borel sets $A, B \in \mathcal{B}(\mathbb{R})$ the two events

$$(X \in A) \text{ and } (Y \in B)$$

are independent. We say that n random variable $X_1, X_2, \cdots, X_n$ are *independent* if for any Borel sets $B_1, B_2, \cdots, B_n \in \mathcal{B}(\mathbb{R})$ the events

$$(X_1 \in B_1),\ (X_2 \in B_2), \cdots, (X_n \in B_n)$$

are independent.

Definition C.10. Two σ-fields $\mathcal{G}$ and $\mathcal{H}$ contain in $\mathcal{F}$ are called *independent* if any two events $A \in \mathcal{G}$ and $B \in \mathcal{H}$ are independent. Similarly, any finite number of σ-fields $\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_n$ contained in $\mathcal{F}$ are *independent* if any n events

$$A_1 \in \mathcal{G}_1, A_2 \in \mathcal{G}_2, \cdots, A_n \in \mathcal{G}_n$$

are independent.

Definition C.11. We say that a random variable X is *independent* of σ-field $\mathcal{G}$ if the σ-fields $\sigma(X)$ and $\mathcal{G}$ are independent.

Definition C.12. A *Stochastic process* is a family of random variable $X(t)$ parametrized by $t \in T$, where $T \subset \mathbb{N}$. When $T = \mathbb{N}$, we shall say that $X(t)$ is a stochastic process in *discrete time* (i.e., a sequence of random variable). When T is an interval in $\mathbb{R}$ (typically $T = [0, \infty)$), we shall say that $X(t)$ is a stochastic process in *continuous time*.

Theorem C.2 (Lebesgue's Dominated Convergence Theorem)**.** Suppose $\{X_n, n \in \mathbb{N}\}$ is a sequence of random variables such that $|X_n| \leq Y$ a.s. where Y is an integrable random variable. If X_n converses to X a.s., then X_n and X are integrable,

$$\lim_{n\to\infty} \int_\Omega X_n d\mathrm{P} = \lim_{n\to\infty} \int_\Omega X d\mathrm{P}$$

and

$$\lim_{n\to\infty} \int_\Omega |X_n - X| d\mathrm{P} = 0.$$

Theorem C.3. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Given a random variable $X : \Omega \to \mathbb{R}$,

$$\int_\Omega g(X(\omega)) d\mathrm{P}(\omega) = \int_\mathbb{R} g(x) d\mathrm{P}_X(x).$$

Theorem C.4. If P_X defined on $\mathbb{R}^n$ is absolutely continuous with density f_X, $g : \mathbb{R}^n \to \mathbb{R}$ is integrable with respect to P_X, then

$$\int_{\mathbb{R}^n} g(x) d\mathrm{P}_X(x) = \int_{\mathbb{R}^n} f_X(x) g(x) dx.$$

Corollary C.5. In the situation of the previous theorem we have

$$\int_{\Omega} g(X) d\mathrm{P} = \int_{\mathbb{R}^n} f_X(x) g(x) dx.$$

Theorem C.6. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. Let X be a real random variable and B a Borel set. Then

$$\int_B g(x) dF_X(x) = \int_{X^{-1}(B)} g(X(\omega)) d\mathrm{P}(\omega).$$

Here g is a Borel function and where $B = \mathbb{R}$

$$\int_{\mathbb{R}} g(x) dF_X(x) = \int_{\Omega} g(X(\omega)) d\mathrm{P}(\omega).$$

Proposition C.7. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space.

(i) F_X is non-decreasing ($y_1 \leq y_2$ implies $F_X(y_1) \leq F_X(y_2)$);

(ii) $\lim_{y \to \infty} F_X(y) = 1, \ \lim_{y \to -\infty} F_X(y) = 0$;

(iii) F_X is right continuous (if $y \to y_0, \ y \geq y_0$, then $F_X(y) \to F_X(y_0)$).

Theorem C.8 (Chebyshev's Inequality)**.** If Y is a non-negative random variable, $\varepsilon > 0$, $0 < p < \infty$, then

$$\mathrm{P}(Y \geq \varepsilon) \leq \frac{\mathrm{E}[Y^p]}{\varepsilon^p}.$$

APPENDIX D

Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and suppose that $\mathcal{G}, \mathcal{H}$ are σ-fields containing in $\mathcal{F}$.

Definition D.1. A random variable $\mathbb{E}[X|\mathcal{G}]$ is called the *conditional expectation* of X relative to a σ-field $\mathcal{G}$ if

(i) $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable,

(ii) $\int_G \mathbb{E}[X|\mathcal{G}] d\mathrm{P} = \int_G X d\mathrm{P}$, for all $G \in \mathcal{G}$.

Proposition D.1. (See Brzeźniak and Zastawniak (1999))
The conditional expectation $\mathbb{E}[X|\mathcal{G}]$ has the following properties:

(i) $\mathrm{E}[\mathbb{E}[X|\mathcal{G}]] = \mathrm{E}[X]$;

(ii) If X is $\mathcal{G}$-measurable, then $\mathbb{E}[X|\mathcal{G}] = X$;

(iii) If X is independent of $\mathcal{F}$, then $\mathbb{E}[X|\mathcal{G}] = \mathrm{E}[X]$
(an independent condition drop out);

(iv) If X is $\mathcal{G}$-measurable and XY is integrable, then $\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}]$
(taking out what is known);

(v) If $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$
(tower property).

Theorem D.2 (Jensen's Inequality)**.** Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function and let X be an integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ such that

$\phi(X)$ is also integrable . Then

$$\phi(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[\phi(X)|\mathcal{G}]$$

for any σ-field $\mathcal{G}$ on Ω contained in $\mathcal{F}$.

Lemma D.3. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. If X is a $\mathcal{G}$-measurable random variable and for any $B \in \mathcal{G}$

$$\int_B X d\mathrm{P} = 0,$$

then $X = 0$ a.s.

Proof. Observe that $\mathrm{P}(X \geq \varepsilon) = 0$ for all $\varepsilon > 0$ because

$$0 \leq \varepsilon \mathrm{P}(X \geq \varepsilon) = \int_{(X \geq \varepsilon)} \varepsilon d\mathrm{P} \leq \int_{(X \geq \varepsilon)} X d\mathrm{P} = 0.$$

Similarly, $\mathrm{P}(X \leq -\varepsilon) = 0$ for all $\varepsilon > 0$. As a consequence,

$$\mathrm{P}(-\varepsilon < X < \varepsilon) = 1$$

for all $\varepsilon > 0$. Let

$$A_n = (-\frac{1}{n} < X < \frac{1}{n}), n \in \mathbb{N}.$$

Then $\mathrm{P}(A_n) = 1$ for all n and

$$(X = 0) = \bigcap_{n=1}^{\infty} A_n.$$

Since $A_n \supset A_{n+1}$ for all n, we obtain

$$\mathrm{P}(X = 0) = \lim_{n \to \infty} \mathrm{P}(A_n) = 1,$$

as required. $\square$

Theorem D.4. Suppose X and Y are $\mathcal{G}$-measurable random variables. If

$$\mathrm{E}[X \mathbb{I}_A] = \mathrm{E}[Y \mathbb{I}_A]$$

for all $A \in \mathcal{G}$, then $X = Y$ a.s.

Proof. From the above lemma, it follows immediately. □

Theorem D.5. Suppose that X and Y are independent and h is a measurable function such that $\mathrm{E}[|h(X,Y)|] < \infty$, then

$$\mathbb{E}[h(X,Y)|\sigma(X)] = g(X)$$

where $g(X(\omega)) = E[h(X(\omega),Y)]$ a.s.

Remark D.1. It is important here that X and Y are independent. This result is not true when X and Y are dependent.

Proof of Theorem D.5. Clearly, $g(X)$ is $\sigma(X)$-measurable. Let $A \in \sigma(X)$, so that there exists a Borel set B such that $A = X^{-1}(B)$. Then

$$\begin{aligned}
\mathrm{E}[h(X,Y)\mathbb{I}_{(X\in B)}] &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x,y)\mathbb{I}_B(x)dF_Y(y)dF_X(x) \\
&= \int_{\mathbb{R}} \mathbb{I}_B(x)\int_{\mathbb{R}} h(x,y)\mathbb{I}_B(x)dF_Y(y)dF_X(x) \\
&= \int_{\mathbb{R}} \mathbb{I}_B(x)\mathrm{E}[h(x,Y)]dF_X(x) \\
&= \mathrm{E}[g(X)\mathbb{I}_{(X\in B)}]. \qquad \text{(D.1)}
\end{aligned}$$

By Definition D.1, we have $g(X) = \mathbb{E}[h(X,Y)|\sigma(X)]$. Consider

$$\begin{aligned}
\int_{\Omega} g(X(\omega))\mathbb{I}_A(\omega)d\mathrm{P}(\omega) &= \int_{\Omega} g(X(\omega))\mathbb{I}_{(X\in B)}(\omega)d\mathrm{P}(\omega) \\
&= \mathrm{E}[g(X)\mathbb{I}_{(X\in B)}] \\
&= \int_{\mathbb{R}} \mathbb{I}_B(x)\mathrm{E}[h(x,Y)]dF_X(x) \\
&= \int_{\Omega} \mathbb{I}_A(\omega)\mathrm{E}[h(X(\omega),Y)]d\mathrm{P}(\omega). \qquad \text{(D.2)}
\end{aligned}$$

From Theorem D.4, we obtain $g(X(\omega)) = \mathrm{E}[h(X(\omega),Y)]$ a.s. □

APPENDIX E

Functional Analysis

We recall some definition and theorem from functional analysis. Most of these results can be found in Kreyszig (1998) and Apostol (1974).

Theorem E.1 (Continuous Mapping)**.** A mapping $T : X \to Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is continuous at a point $x_0 \in X$ if and only if

$$x_n \to x_0 \text{ implies } Tx_n \to Tx_0.$$

Definition E.1. A metric space X is said to be *compact* if every sequence in X has a convergent subsequence. A subset M of X is said to be *compact* if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element in M.

Theorem E.2 (Compactness)**.** In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded.

Theorem E.3 (Continuous Mapping)**.** Let X and Y be metric spaces and $T : X \to Y$. Then the image of a compact subset M of X under T is compact.

Corollary E.4 (Maximum and Minimum)**.** A continuous mapping T of a compact subset M of a metric space X into $\mathbb{R}$ assumes a maximum and a minimum at some points of M.

Theorem E.5 (Canter Intersection Theorem)**.**
Let $\{F_1, F_2, F_3, \cdots\}$ be a countable collection of nonempty sets in $\mathbb{R}^m$ such that:

(i) $F_{n+1} \subset F_n,\ n \in \mathbb{N}$;

(ii) each set F_n is closed and F_1 is bounded. Then the intersection $\cap_{n=1}^{\infty} F_n$ is closed and nonempty.

Theorem E.6 (Bolzano's Theorem)**.**

Assume f is real-valued and continuous on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs; that is, assume $f(a)f(b) < 0$. Then there is at least one point c in the open interval (a, b) such that $f(c) = 0$.

Definition E.2 (Orthogonality)**.** An element x of an inner product space X is said to be *orthogonal* to an element $y \in X$ if

$$\langle x, y \rangle = 0.$$

Theorem E.7 (Schwarz Inequality)**.** An inner product and the corresponding norm satisfy the Schwarz inequality, i.e.,

$$|\langle x, y \rangle| \leq \|x\| \|y\|.$$

Theorem E.8 (Minimizing Vector)**.** Let X be an inner product space and $M \neq \emptyset$ a convex subset which is complete (in the metric induced by the inner product). Then for every given $x \in X$ there exists a unique $y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\| = \|x - y\|.$$

Theorem E.9 (Orthogonality)**.** In theorem E.8, let M be a complete subspace Y and $x \in X$ fixed. Then $z = x - y$ is orthogonal to Y.

Definition E.3 (Direct Sum)**.** A vector space X is said to be the *direct sum* of two subspaces of Y and Z , written

$$X = Y \oplus Z,$$

if each $x \in X$ has a unique representation

$$x = y + z$$

for some $y \in Y$ and $z \in Z$. Then Z is called *algebraic complement* of Y in X and vice versa, and Y, Z is called a *complement pair* of subspaces in X.

In the case of a general Hilbert space H, the main interest concerns representations of H as a direct sum of a closed subspace Y and its orthogonal complement

$$Y^{\perp} := \{z \in H : z \perp Y\},$$

which is the set of all vectors orthogonal to Y.

Theorem E.10 (Direct Sum)**.** Let Y be any closed subset of a Hilbert space H. Then $H = Y \oplus Z$ and $Y \cap Z = \{0\}$ when $Z = Y^{\perp}$.

In theorem E.10, we found that for every $x \in H$ there exists and unique a $y \in Y$ and $z \in Y^{\perp}$ such that $x = y + z$, y is called the *orthogonal projection* of x on Y. Define a mapping

$$\begin{aligned} \rho : \quad H &\rightarrow Y \\ x &\mapsto y = \rho(x). \end{aligned}$$

ρ is called the (*orthogonal*) *projection* of H onto Y.