A GEOMETRIC BROWNIAN MOTION MODEL WITH COMPOUND POISSON PROCESS AND FRACTIONAL STOCHASTIC VOLATILITY

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Abstract

In this paper, we introduce an approximate approach to a geometric Brownian motion (gBm) model with compound Poisson processes and fractional stochastic volatility. Based on a fundamental result on the \( L^2 \)-approximation of this fractional noise by semimartingales, we prove a convergence theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a gBm with jump and fractional stochastic volatility as compared to the stochastic volatility.

1. Introduction

Let \((\Omega, \mathcal{F}, P)\) be a probability space and all the processes that we shall consider in this paper will be defined in this space. Then a geometric Brownian motion model (or a diffusion model) is a model of the form

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\[
dS_t = S_t(\mu \, dt + \sigma \, dW_t), \quad t \in [0, T] \quad \text{and} \quad T < \infty,
\]

where \( \mu \in \mathbb{R}, \sigma > 0, \quad S = (S_t)_{t \in [0,T]} \) is a process representing the price of the underlying assets, and \((W_t)_{t \in [0,T]}\) is the standard Brownian motion.

Over the last decade, many academic researchers have tried to extend and improve the classical geometric Brownian motion model in various directions. Some researchers represent rare events by jumps and introduce a model of jump diffusion (see Merton [1] and Kou [2]). Other authors try to provide a more realistic stochastic process for the underlying process (e.g., stock price) by introducing a stochastic process for the volatility, i.e., with the variance of the stock return as random. See, for example, Hull and White [3], Stein and Stein [4] and Heston [5]. Since there is an empirical study showing that the behavior of stock price exhibits a long-range dependence, Thao [6] replaced Brownian motions (Bm) by fractional Brownian motions (fBm) in the diffusion model. Moreover, Sattayatham et al. [7] extended Thao’s results by adding a Poisson jump into the model. In this paper, we shall extend our investigations by replacing a Poisson jump by a compound Poisson jump and assuming that the variance of the stock return follows a fractional stochastic process.

The paper is organized as follows: In Section 2, we recall some preliminary properties of an approximate approach to fBm. A jump measure in the compound Poisson process is reviewed in Section 3. In Section 4, the geometric Brownian motion model with a compound Poisson process and fractional stochastic volatility is introduced. The convergent theorem of the approximate solution to the limit process is established in Section 5. Finally, we give some simulation examples to show the accuracy of the approximations by the geometric Brownian motion with stochastic volatility as compared to the geometric Brownian motion with fractional stochastic volatility.

### 2. An Approximate Approach to fBm

In this section, we shall review definitions of a fBm and its approximation by a semimartingale. Details of the discussion can be found in Thao [6].

A fractional Brownian motion with Hurst \( H \in (0, 1) \) is a centered Gaussian process \( B^H = (B^H_t)_{t \geq 0} \) with zero mean, and the covariance function is given by

\[
R(t, s) = E(B^H_t B^H_s) = \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}).
\]
This process was introduced by Kolmogorov [8] and studied further by Mandelbrot and Van Ness [9].

If $H = 1/2$, then $R(t, s) = \min(t, s)$ corresponds to the ordinary Brownian motion. The fractional Brownian motion $B^H$ is neither a semimartingale nor a Markov process in the case $H \neq 1/2$.

The fractional Brownian motion $B^H$ exhibits a long-range dependence in the sense that the infinite series of $\rho_n = E[B^H_1 (B^H_{n+1} - B^H_n)]$ is either divergent or convergent with very late rate.

Furthermore, it has a stationary increment, i.e., the increment of $B^H$ has a normal distribution with zero mean and its variance is given by

$$E[(B^H_t - B^H_s)^2] = |t - s|^{2H},$$

in an interval $[s, t]$.

It is well known that $B^H$ admits the following representation:

$$B^H_t = \frac{1}{\Gamma(1 - \alpha)} \left\{ Z_t + \int_0^t (t - s)^\alpha dW_s \right\},$$

where $W = (W_s)_{s \geq 0}$ is a standard Brownian motion $\alpha = H - \frac{1}{2}$, $H \in (0, 1)$ and

$$Z_t = \int_{-\infty}^0 [(t - s)^\alpha - (s)^\alpha] dW_s.$$  

Instead of $B^H$, Alòs et al. [10] proposed using

$$B_t = \int_0^t (t - s)^\alpha dW_s$$  

(2)

since the process $Z_t$ has absolutely continuous trajectories, so it suffices to consider only the term $B_t$, that has a long-range dependence. Note that $B_t$ can be approximated by

$$B^e_t = \int_0^t (t - s + \epsilon)^\alpha dW_s$$  

(3)
in the sense that $B_i^\varepsilon$ converges to $B_i$ in $L_p(\Omega)$ as $\varepsilon \to 0$ for any $p \geq 2$, uniform with respect to $t \in [0, T]$ (see, Dung [11]). Since $(B_i^\varepsilon)_{\varepsilon \in [0, T]}$ is a continuous semimartingale, Ito calculus can be applied to the following SDE:

$$dS_i^\varepsilon = S_i^\varepsilon (\mu dt + \sigma dB_i^\varepsilon), \quad 0 \leq t \leq T.$$ 

Let $S_i^\varepsilon$ be the solution of the above equation. Because of the convergence of $B_i^\varepsilon$ to $B_i$ in $L^2(\Omega)$ when $\varepsilon \to 0$, we shall define a solution of a fractional stochastic differential equation of the form

$$dS_i = S_i (\mu dt + \sigma dB_i), \quad 0 \leq t \leq T,$$

to be a process $S_i^\varepsilon$ defined on the probability space $(\Omega, \mathcal{F}, P)$ such that the process $S_i^\varepsilon$ converges to $S_i^\varepsilon$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and the convergence is uniform with respect to $t \in [0, T]$. This definition will be applied to the other similar fractional stochastic differential equations which will appear later.

### 3. Compound Poisson Processes

In this section, we shall review the notion of the compound Poisson process and some of its properties which will be useful in the sequel; see Cont and Tankov [12, pp. 57-78] for the details.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $E \subseteq \mathbb{R}^d$ and $\mu$ as a (positive) Radon measure on $(E, \mathcal{E})$. Then a Poisson random measure on $E$ with intensity measure $\mu$ is an integer-valued random measure $M : \Omega \times \mathcal{E} \to \mathbb{N}$ such that:

- For (almost all) $\omega \in \Omega$, $M(\omega, \cdot)$ is an integer-valued Radon measure on $E$ and, for any bounded measurable set $A \subseteq E$, $M(\cdot, A) := M(A) < \infty$ is an integer-valued random variable.

- For each measurable set $A \subseteq E$, $M(A)$ is a Poisson random variable with parameter $\mu(A)$ such that

$$\forall k \in \mathbb{N}, \quad P(M(A) = k) = e^{-\mu(A)} \frac{(\mu(A))^k}{k!}.$$
For disjoint measurable sets $A_1, \ldots, A_n \subset \mathcal{E}$, the random variables $M(A_1), \ldots, M(A_n)$ are independent.

We can prove that for any Radon measure $\mu$ on $E \subset \mathbb{R}^d$, there exists a Poisson random measure $M$ on $E$ with intensity $\mu$. Consequently, any Poisson random measure on $E$ can be represented as a counting measure associated with a random sequence of points in $E$, i.e., there exists $(T_n(\omega))_{n \geq 1}$, such that

$$\forall A \in \mathcal{E}, \quad M(\omega, A) = \sum_{n \geq 1} 1_A(T_n(\omega)) = \#\{i \geq 1, T_i(\omega) \in A\}. \quad (4)$$

Define a random variable $T_n = \sum_{i=1}^n \tau_i$, where $(\tau_i)_{i \geq 1}$ is a sequence of independent exponential random variables with parameter $\lambda$, that is, $P(\tau_i > t) = e^{-\lambda t}$. The process $(N_t)_{t \geq 0}$, defined by

$$N_t = \sum_{n \geq 1} 1_{t > T_n}$$

is called a Poisson process with intensity $\lambda$.

Moreover, by equation (4), the Poisson process may be expressed in terms of the Poisson random measure $M$ in the following way:

$$N_t(\omega) = M(\omega, [0, t]) = \int_{[0,t]} M(\omega, ds),$$

where $ds$ is the Lebesgue area element on $[0, t]$.

A compound Poisson process on $\mathbb{R}^d$ with intensity $\lambda > 0$ and jump size distribution $f$ is a stochastic process $X_t$ defined as

$$X_t = \sum_{i=1}^{N_t} Y_i,$$

where jump sizes $Y_i$ are independent and identically distributed (i.i.d.) with distribution $f$ and $(N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda$, independent from $(Y_i)_{i \geq 1}$. The Poisson process itself can be seen as a compound Poisson process on $\mathbb{R}$ such that $Y_i = 1$. This explains the origin of the term “compound Poisson” in the definition.
For every compound Poisson process \((X_t)_{t \geq 0}\) on \(\mathbb{R}^d\) with intensity \(\lambda\) and jump size distribution \(f\), its jump measure
\[
J_X(B) = \#\{(X_t - X_{t-}, t) \in B\}
\]
is a Poisson random measure on \(\mathbb{R}^d \times [0, \infty)\) with intensity measure \(\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt\), where \(B\) is a measurable subset of \(\mathbb{R}^d \times [0, \infty)\) and \(\nu\) is the Lévy measure of the compound Poisson process.

This fact implies that every compound Poisson process can be represented in the following form:
\[
X_t = \sum_{s \in [0,t]} \Delta X_s = \int_{\mathbb{R}^d \times [0,t]} xJ_X(dx \times ds),
\]
where \(J_X\) is a Poisson random measure with intensity measure \(\nu(dx)dt\).

Let \(E\) be a measurable subset of \(\mathbb{R}^d\). Then for a measurable function \(f : E \times [0, T] \to \mathbb{R}^d\), we can construct an integral with respect to the Poisson random measure \(M\). It is given by the random variable
\[
\int_{E \times [0,T]} f(y, s)M(\cdot, dx \times ds) = \sum_{n \geq 1} f(Y_n(\cdot), T_n(\cdot)).
\]

4. A gBm with Compound Poisson Jumps and Fractional Stochastic Volatility

Suppose \((\Omega, \mathcal{F}, P)\) is a probability space on which we define two standard Brownian motions \((W_t)_{t \geq 0}\) and \((\overline{W}_t)_{t \geq 0}\) with correlation \(\rho\) and a compound Poisson process \((Z_t)_{t \geq 0}\) with intensity \(\lambda\) and Gaussian distribution of jump sizes.

We assume that the \(\sigma\)-algebras generated, respectively, by \((Z_t)_{t \geq 0}\), \((W_t)_{t \geq 0}\) and \((\overline{W}_t)_{t \geq 0}\) are independent.

Suppose that a single stock price \(S_t\) and its volatility \(\nu_t = \sigma_t^2\) satisfy the following stochastic differential equations:
\[
dS_t = S_t(\mu dt + \sqrt{\nu_t}dW_t) + S_t_- dZ_t, \tag{5}
\]
\[
d\nu_t = (\omega - \theta \nu_t)dt + \xi \nu_t d\overline{W}_t \tag{6}
\]
with initial conditions $S_{t(t=0)} = S_0 \in L_2(\Omega)$ and $v_{t(t=0)} = v_0 \in L_2(\Omega)$, where

$\mu$ is the (deterministic) instantaneous drift of stock price returns,

$\omega$ is the mean long-term volatility,

$\theta$ is the rate at which the volatility reverts toward its long-term mean,

$\xi$ is the volatility of the volatility process.

The notation $S_{t-}$ means that whenever there is a jump, the value of the process before the jump is used on the left-hand side of the formula. The last term of equation (5) is just a symbol. More precisely, it can be defined by a stochastic integral with respect to the Poisson random measure $N(\omega, \cdot)$ as the sum of jumps of a Poisson process $N_t$,

$$\int_0^t S_{s-}dZ_s = \int_0^t Y_s S_{s-}dN_s = \sum_{i=1}^{N_t} \Delta S_{T_i}, \tag{7}$$

where $Y_t$ is a random jump amplitude.

We now assume that the $T_i$'s correspond to the jump times of a Poisson process $N_t$ and that $Y_i$ is a sequence of identically distributed random variables with values in $(-1, \infty)$. Let $S_t$ be a predictable process. Then at time $T_i$, the jump of the dynamics of $S_t$ is given by

$$\Delta S_{T_i} := S(T_i) - S(T_i-) = Y_i S(T_i-)$$

which, by the assumption $Y_i > -1$, leads always to positive values of the prices.

To solve equation (5), let us rewrite it into an integral form as follows:

$$S_t = S_0 + \int_0^t \mu S_s ds + \int_0^t \sqrt{v_s} S_s dW_s + \sum_{i=1}^{N_t} \Delta S_{T_i}. \tag{8}$$

Assume $\mathbb{E}\left[\int_0^T v_s^2 dt\right] < \infty$. Then, by an application of Ito’s lemma for the jump-diffusion process (Cont and Tankov [12, p. 275]) on equation (8) with $f(S_t, t) = \log(S_t)$, we get

$$\log S_t = \log S_0 + \mu t - \frac{1}{2} \int_0^t v_s^2 ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1 + Y_s) dN_s,$$
or, equivalently,

\[ S_t = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \nu_s ds + \int_0^t \sqrt{\nu_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right]. \]

Since in many problems related to network traffic analysis, mathematical finance, and many other fields, the processes under study seem empirically to exhibit long-range dependent properties, their dynamics should be driven by a fractional Brownian process. Hence, instead of (6), we consider the fractional version of (6):

\[ dS_t = S_t (\mu dt + \sqrt{\nu_t} dW_t) + S_t Y_t dN_t, \quad (9) \]

\[ dv_t = (\omega - \theta v_t) dt + \xi v_t dB_t, \quad (10) \]

where \( B_t \) is as given in equation (2). The corresponding approximately fractional model can be defined, for each \( \varepsilon > 0 \), by

\[ dS_t^\varepsilon = S_t^\varepsilon (\mu dt + \sqrt{\nu_t^\varepsilon} dW_t) + S_t^\varepsilon Y_t dN_t, \quad (11) \]

\[ dv_t^\varepsilon = (\omega - \theta v_t^\varepsilon) dt + \xi v_t^\varepsilon dB_t^\varepsilon, \quad (12) \]

where \( B_t^\varepsilon \) is as given in equation (3).

In the paper of Plienpanich et al. [14], the solution of the approximate model (12) with initial condition \( v_{t(0)} = v_0 \in L^2(\Omega) \) is given by

\[ v_t^\varepsilon = \left( v_0 + \omega \int_0^t \exp(\kappa s - \xi B_t^\varepsilon) ds \right) \exp(\xi B_t^\varepsilon - \kappa t), \quad (13) \]

where \( \kappa = 0 + \frac{1}{2} \xi^2 e^{2\alpha} \), \( \varepsilon > 0 \), \( \alpha \in (0, 1/2) \), and \( \theta, \xi \) are real constants.

Assume that \( E \left[ \int_0^T v_t^\varepsilon (S_t^\varepsilon)^2 dt \right] < \infty \). Using Ito’s formula for the jump process again, the solution of the approximate model (11) is given by

\[ \log S_t^\varepsilon = \log S_0 + \mu t - \frac{1}{2} \int_0^t \nu_s ds + \int_0^t \sqrt{\nu_s} dW_s + \int_0^t \log(1 + Y_s) dN_s, \]

or, equivalently,

\[ S_t^\varepsilon = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \nu_s ds + \int_0^t \sqrt{\nu_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right]. \quad (14) \]
5. Convergence of a Solution of an Approximate Model

Before stating the main theorem concerning the convergence $S_t^\varepsilon$ to a random variable $S_t^\varepsilon \in L_2(\Omega)$ as $\varepsilon \to 0$, we first prove the convergence of the process $\{v_t^\varepsilon : \varepsilon > 0\}$ in $L_2(\Omega)$ as $\varepsilon \to 0$. Denoting the norm in $L_2(\Omega)$ by $\| \cdot \|_2$:

**Lemma 1.** Let $p, q, r \in [1, \infty)$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Suppose that $v_0(\cdot)$ is a random variable such that $E|v_0|^p < \infty$. Then $\|v_t^\varepsilon\|_r < \infty$ for all $t \in [0, T]$.

**Proof.** Using the fact that $\|f/g\|_r \leq \|f\|_p \|g\|_q$, where $p, q, r \in [1, \infty)$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ (see Jones [13, p. 226]), we have

$$\|v_t^\varepsilon\|_r = \left\| v_0 + \omega \int_0^t \exp(\kappa s - \xi B_t^\varepsilon) ds \exp(\xi B_t^\varepsilon - \kappa t) \right\|_r$$

$$\leq \|v_0\|_p \|\exp(\xi B_t^\varepsilon - \kappa t)\|_r + |\omega| \left\| \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) ds \exp(\xi B_t^\varepsilon - \kappa t) \right\|_r$$

$$\leq \|v_0\|_p \|\exp(\xi B_t^\varepsilon - \kappa t)\|_r + |\omega| \left\| \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) ds \right\|_p \|\exp(\xi B_t^\varepsilon - \kappa t)\|_q$$

$$\leq \|v_0\|_p \|\exp(\xi B_t^\varepsilon - \kappa t)\|_q + |\omega| \left\| \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) ds \right\|_p \|\exp(\xi B_t^\varepsilon - \kappa t)\|_q$$

$$\leq (E|v_0|^p)^{1/p} \exp(-\kappa t) e^{q^2\gamma^2_\varepsilon(t)/2} + |\omega| t \exp(\kappa t) e^{p^2\gamma^2_\varepsilon(t)/2} \exp(-\kappa t) e^{q^2\gamma^2_\varepsilon(t)/2}$$

$$< \infty, \text{ for all } t \in [0, T].$$

For example, we shall compute $\|\exp(\xi B_t^\varepsilon - \kappa t)\|_q$. Note that $B_t^\varepsilon$ is a Gaussian process with null means and finite variance. Let $\gamma^2_\varepsilon(t)$ be the variance of $B_t^\varepsilon$. Then we get $\gamma^2_\varepsilon(t) = E|B_t^\varepsilon|^2 = \frac{(t + \varepsilon)^{2\alpha+1} - e^{2\alpha+1}}{2\alpha + 1}$ (see Dung [11]). Hence
\[ \| \exp(\xi B_t^\varepsilon - \kappa t) \|_q \]
\[ = \exp(-\kappa t) [E| e^{\xi B_t^\varepsilon}|^q]^{1/q} \]
\[ = \exp(-\kappa t) [E(e^{q\xi B_t^\varepsilon})]^{1/q} \]
\[ = \exp(-\kappa t) \left[ \frac{1}{\sqrt{2\pi \gamma(t)}} \int_{-\infty}^{\infty} e^{\gamma(t) / 2} e^{\frac{-z^2}{2 \gamma(t)}} dz \right]^{1/q} \]
\[ = \exp(-\kappa t) \left[ \frac{1}{\sqrt{2\pi \gamma(t)}} \int_{-\infty}^{\infty} e^{q^2 \xi^2 / 2 \gamma(t)} e e^{\frac{-z^2}{2 \gamma(t)}} dz \right]^{1/q} \]
\[ = \exp(-\kappa t) e^{q^2 \xi^2 / 2 \gamma(t)} < \infty, \quad \text{for all} \ t \in [0, T]. \]

The other expressions can be computed similarly. This proves the lemma.

\textbf{Lemma 2.} Let \( r \in [1, \infty) \) and \( p, q \geq 2 \) with \( 1 = \frac{1}{r} + \frac{1}{q} \) and suppose that \( v_0(\cdot) \) is a random variable such that \( E|v_0|_p^r < \infty \). Then the process \( \{v_t^\varepsilon : \varepsilon > 0\} \) converges to \( v_t \) in \( L_r(\Omega) \) as \( \varepsilon \to 0 \). This convergence is uniform with respect to \( t \in [0, T] \).

\textbf{Proof.} Define a process \( \{v_t : t \in [0, T]\} \) as follows:
\[ v_t = \left( v_0 + \omega \int_0^t \exp(\kappa s - \xi B_s) ds \right) \exp(\xi B_t - \kappa t), \]
where all the parameters are defined the same as in equation (13).
Using the fact that $B_t$ is a Gaussian process with null means and finite variance and its variance $\gamma_t^2 := E[B(t)]^2 = \frac{t^{2\alpha+1}}{2\alpha+1}$, as in Lemma 1, we can show that $\nu_t \in L_p(\Omega)$. Next we compute

$$v_t^\varepsilon - v_t = \left[ v_0 + \omega \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \right] \exp(\xi B_t^\varepsilon - \kappa t)$$

$$- \left[ v_0 + \omega \int_0^t \exp(\kappa s - \xi B_s) \, ds \right] \exp(\xi B_t - \kappa t)$$

$$= v_0 \left[ \exp(\xi B_t^\varepsilon - \kappa t) - \exp(\xi B_t - \kappa t) \right]$$

$$+ \omega \left[ \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \exp(\xi B_t^\varepsilon - \kappa t) \right]$$

$$- \int_0^t \exp(\kappa s - \xi B_s) \, ds \exp(\xi B_t - \kappa t) \right].$$

The second expression of the last equation is equal to

$$\omega \left( \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \exp(\xi B_t^\varepsilon - \kappa t) - \int_0^t \exp(\kappa s - \xi B_s) \, ds \exp(\xi B_t - \kappa t) \right)$$

$$= \omega \left( \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \exp(\xi B_t^\varepsilon - \kappa t) - \int_0^t \exp(\kappa s - \xi B_s) \, ds \exp(\xi B_t - \kappa t) \right)$$

$$+ \omega \left[ \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \exp(\xi B_t - \kappa t) - \int_0^t \exp(\kappa s - \xi B_s) \, ds \exp(\xi B_t - \kappa t) \right]$$

$$= \omega \left[ \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \right] \left[ (\exp(\xi B_t - \kappa t))(\exp(\xi (B_t^\varepsilon - B_t)) - 1) \right]$$

$$+ \omega \left[ \int_0^t (\exp(\kappa s - \xi B_s^\varepsilon))(\exp(\xi (B_s - B_s^\varepsilon)) - 1) \, ds \exp(\xi B_t - \kappa t) \right].$$

Consequently,

$$v_t^\varepsilon - v_t = v_0 \exp(\xi B_t - \kappa t) \left[ \exp(\xi (B_t^\varepsilon - B_t)) - 1 \right]$$

$$+ \omega \left[ \int_0^t \exp(\kappa s - \xi B_s^\varepsilon) \, ds \right] \left[ (\exp(\xi B_t - \kappa t))(\exp(\xi (B_t^\varepsilon - B_t)) - 1) \right]$$

$$+ \omega \left[ \int_0^t (\exp(\kappa s - \xi B_s^\varepsilon))(\exp(\xi (B_s - B_s^\varepsilon)) - 1) \, ds \exp(\xi B_t - \kappa t) \right].$$
Hence,

\[ \| v_t^\varepsilon - v_t \|_r \leq \| v_0 \exp(\xi B_t - \kappa t) [\exp(\xi (B_t^\varepsilon - B_t)) - 1] \|_r \]

\[ + \left\| \omega \int_0^t \exp(ks - \xi B_s^\varepsilon) ds \right\| \left\| (\exp(\xi B_t^\varepsilon - t) - 1) \right\|_r \]

\[ + \left\| \omega \int_0^t (\exp(ks - \xi B_s)) [\exp(\xi (B_s^\varepsilon - B_s)) - 1] ds \exp(\xi B_t - \kappa t) \right\|_r. \]

(15)

We aim to prove that \( \| v_t^\varepsilon - v_t \|_r \to 0 \) in \( L_r(\Omega) \) as \( \varepsilon \to 0 \). To do this we note that

\( \| v_0 \|_p = (E|v_0|^p)^{1/2} < \infty \) and, recalling that \( B_t \) is the Gaussian process with null means and finite variance \( \gamma_t^2 \), then

\[ \| \exp(\xi B_t - \kappa t) \|_q = \exp(-\kappa t) \left[ E \exp(\xi B_t^2) \right]^{1/2} \]

\[ = \exp(-\kappa t) \left[ \frac{1}{\sqrt{2\pi \gamma_t^2}} \int_{-\infty}^{\infty} e^{\frac{-(z^2)}{2\gamma_t^2}} dz \right]^{1/2} \]

\[ = \exp(-\kappa t) \left[ \frac{1}{\sqrt{2\pi \gamma_t^2}} \int_{-\infty}^{\infty} e^{\frac{- z^2 - 4q_z^2 \gamma_t^2 z + 4q_z^2 \gamma_t^2 z^2}{2\gamma_t^2}} \right]^{1/2} \]

\[ = \exp(-\kappa t) e^{\frac{z^2}{2\gamma_t^2}} \left[ \frac{1}{\sqrt{2\pi \gamma_t^2}} \int_{-\infty}^{\infty} e^{\frac{- (z-2q_z \gamma_t)^2}{2\gamma_t^2}} \right]^{1/2} \]

\[ = \exp(-\kappa t) e^{\frac{z^2}{2\gamma_t^2}} < \infty, \quad \text{for all } t \in [0, T]. \]
Next, it follows from the relation\( e^{\| A \|_{q} 1 - 1 = \| A \|_{2q} + o(\| A \|_{2q})} \) that
\[
\| (\exp(\xi(B_t^e - B_t)) - 1) \|_{2q} = \| \xi \| \| (B_t^e - B_t) \|_{2q} + o(\| \xi(B_t^e - B_t) \|_{2q}).
\]
(17)
Since \( \| (B_t^e - B_t) \|_{2q} \to 0 \) as \( \epsilon \to 0 \), equation (17), and the first expression of equation (15) approaches zero as \( \epsilon \to 0 \).

For the second expression of (15), we note that
\[
\left\| \int_0^t \exp(\xi_s - \xi B_s^e) ds \right\|_p \leq \int_0^t \left\| \exp(\xi_s - \xi B_s^e) \right\|_p ds = \epsilon \| \exp(\epsilon t) \| e^{\epsilon^2 t^2 / 2} < \infty, \quad \text{for all } t \in [0, T].
\]
Since (17) approaches zero as \( \epsilon \to 0 \), the second expression of equation (15) approaches zero as \( \epsilon \to 0 \).

Finally, for the third expression of (15), we have
\[
\left\| \int_0^t \left[ \exp(\xi_s - \xi B_s^e) \right] \left[ \exp(\xi(B_s - B_s^e)) - 1 \right] ds \right\|_p
\]
\[
\leq \int_0^t \| (\exp(\xi_s - \xi B_s^e)) \|_2 \| (\exp(\xi(B_s - B_s^e)) - 1) \|_2 ds
\]
\[
\leq \int_0^t \| (\exp(\xi_s - \xi B_s^e)) \|_2 \| (\exp(\xi(B_s - B_s^e)) - 1) \|_2 ds
\]
\[
= \| \xi \| \| (B_t^e - B_t) \|_{2q} + o(\| (\xi(B_s - B_s^e)) \|_{2p}) \int_0^t \| (\exp(\xi_s - \xi B_s^e)) \|_2 ds
\]
\[
\leq \epsilon \| \exp(\epsilon t) \| e^{\epsilon^2 t^2 / 2} \| (B_t^e - B_t) \|_{2p} + o(\| (\xi(B_s - B_s^e)) \|_{2p}) \to 0
\]
as \( \epsilon \to 0 \). Thus, the third expression of equation (15) approaches zero as \( \epsilon \to 0 \).

Consequently, all expressions of the right hand side of equation (15) approach zero as \( \epsilon \to 0 \). Therefore, \( v_t^e \to v_t \) in \( L_p(\Omega) \) as \( \epsilon \to 0 \) and this convergence does not depend on \( t \), i.e., the convergence is uniform with respect to \( t \in [0, T] \). \( \square \)
Now we ready to state and prove our main results. The solution of the approximated model (11) is given by

\[ S_t^\varepsilon = S_0 \exp \left( \mu - \frac{1}{2} \int_0^t v_s^\varepsilon ds + \int_0^t \sqrt{v_s^\varepsilon} dW_s + \int_0^t \log(1 + Y_s) dN_s \right). \]  

(18)

Define a stochastic process \( S_t^* \) as follows:

\[ S_t^* = S_0 \exp \left( \mu - \frac{1}{2} \int_0^t v_s ds + \int_0^t \sqrt{v_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right). \]  

(19)

The following theorem shows that the process \( S_t^* \) is the limit process of \( S_t^\varepsilon \) in \( L^2(\Omega) \) as \( \varepsilon \to 0 \). Hence, by definition, \( S_t^* \) will be the solution of equation (9).

**Theorem 3.** Suppose that \( S_0(\cdot) \) is a random variable such that \( E|S_0|^4 \) is finite and the initial condition \( v_0(\cdot) \neq 0 \). The stochastic process \( S_t^\varepsilon \) of (18) converges to the limit process \( S_t^* \) in \( L^2(\Omega) \) as \( \varepsilon \to 0 \) and the convergence is uniform with respect to \( t \in [0, T] \) with \( 0 < \alpha < 1/2 \).

**Proof.** It follows from equations (18) and (19) that

\[ S_t^\varepsilon - S_t^* = S_0 \left( \frac{S_t^*}{S_0} \right) \left[ \exp \left( -\frac{1}{2} \int_0^t (v_s^\varepsilon - v_s) ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s}) dW_s \right) - 1 \right]. \]

Thus

\[
\| S_t^\varepsilon - S_t^* \|_2 \\
\leq \| S_0 \|_4 \left\| \left( \frac{S_t^*}{S_0} \right) \left[ \exp \left( -\frac{1}{2} \int_0^t (v_s^\varepsilon - v_s) ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s}) dW_s \right) - 1 \right] \right\|_4 \\
\leq \| S_0 \|_4 \left\| \frac{S_t^*}{S_0} \right\|_8 \left\| \left[ \exp \left( -\frac{1}{2} \int_0^t (v_s^\varepsilon - v_s) ds + \int_0^t (\sqrt{v_s^\varepsilon} - \sqrt{v_s}) dW_s \right) - 1 \right] \right\|_8.
\]

(20)

The following three parts show an approximation of norm \( \| S_t^\varepsilon - S_t^* \|_2 \) in equation (20):
• $S_0 \mathbb{E}^t = \mathbb{E}[S_0] = E[S_0] \mathbb{E} < \infty$ by the assumptions of the theorem.

• We note that

$$\left\| \frac{S_t^*}{S_0} \right\|_8 \leq \exp\left(\mu + \int_0^t \log(1 + Y_s) dN_s\right) \times \exp\left(-\frac{1}{2} \int_0^t v_s ds\right) \times \exp\left(\int_0^t \sqrt{v_s} dW_s\right)$$

$$\leq \exp\left(\mu + \int_0^t \log(1 + Y_s) dN_s\right) \times \exp\left(-\frac{1}{2} \int_0^t v_s ds\right) \times \exp\left(\int_0^t \sqrt{v_s} dW_s\right)$$

$$\times \exp\left(\int_0^t \sqrt{v_s} dW_s\right) \leq K \exp(|\mu| T),$$

where $K$ is a constant. The last inequality follows from the fact that there are a finite number of jumps in the finite interval $[0, T]$.

Moreover,

$$\left\| \exp\left(-\frac{1}{2} \int_0^t v_s ds\right) \right\|_{32} \leq \exp\left(-\frac{1}{2} \int_0^t v_s ds\right) \leq \exp\left(\frac{1}{2} M_t T\right) < \infty,$$
where $M_1 := \sup_{0 \leq t \leq T} \| v_t \|_{32}$. The maximum exists since $v_t \in L_{32}(\Omega)$ (by Lemma 2) and

\[ \| v_t \|_{32}^3 = E \left| \exp \left( (\xi \beta_t - k t) \left( v_0 + \omega \int_0^t \exp(\kappa s - \xi \beta_t) \, ds \right) \right) \right| \]

is continuous with respect to $t \in [0, T]$.

For the remaining term, we note that

\[ \left\| \exp \left( \int_0^t \sqrt{v_s} \, dW_s \right) \right\|_{32}^3 \leq \exp \left( \left\| \int_0^t \sqrt{v_s} \, dW_s \right\|_{32}^3 \right) \leq \exp(M_2 \| W_t - W_0 \|_{32}) = \exp(M_2 M_3) < \infty, \]

where $M_2 := \sup_{0 \leq t \leq T} \| \sqrt{v_t} \|_{32}$ and $M_3 := \sup_{0 \leq t \leq T} \| W_t \|_{32}$. The maximum exists since

\[ \| W_t \|_{32}^3 = E|W_t|^3 = \frac{(32)!}{2^{16} \cdot 16!} t^{16} < \infty \]

and is continuous with respect to $t \in [0, T]$.

Consequently, we see that $\left\| S^*_t \right\|_8$ is finite.

- The third term on the right hand side of equation (20) is calculated by using a relation $\exp(A) - 1 = A + o(A)$. So we have

\[ \left\| \exp \left( -\frac{1}{2} \int_0^t (v_s^e - v_s) \, ds + \int_0^t (\sqrt{v_s^e} - \sqrt{v_s}) \, dW_s \right) - 1 \right\|_8 \]

\[ \leq \frac{1}{2} \int_0^t (v_s^e - v_s) \, ds + \int_0^t (\sqrt{v_s^e} - \sqrt{v_s}) \, dW_s \] + $R$

\[ \leq \frac{1}{2} \int_0^t \| v_s^e - v_s \|_8 \, ds + \int_0^t \frac{\| v_s^e - v_s \|_8}{\sqrt{v_s^e} + \sqrt{v_s}} \, dW_s + R$

\[ \leq \frac{1}{2} \int_0^t \| v_s^e - v_s \|_8 + \frac{\| v_s^e - v_s \|_8}{\sqrt{v_s^e} + \sqrt{v_s}} \| W_t - W_0 \|_8 + R. \]
where

$$\bar{R} = \left\| \left( -\frac{1}{2} \int_0^t (v_s^e - v_s) \, ds + \int_0^t (\sqrt{v_s^e} - \sqrt{v_s}) \, dW_s \right)^\epsilon \right\|_8.$$  

Hence,

$$\left\| \left[ \exp \left( -\frac{1}{2} \int_0^t (v_s^e - v_s) \, ds + \int_0^t (\sqrt{v_s^e} - \sqrt{v_s}) \, dW_s \right) - 1 \right] \right\|_8 \leq \frac{1}{2} \| v_s^e - v_s \|_8 + \frac{\| v_s^e - v_s \|_8}{\| \sqrt{v_s^e} + \sqrt{v_s} \|_8} \hat{M} + \bar{R},$$  

\hspace*{1in} \hspace*{1in} (22)

where \( \hat{M} := \max_{0 \leq s \leq T} \| W_s \|_8 \).

Note that \( 0 < c \leq \| \sqrt{v_s^e} + \sqrt{v_s} \|_8 \) for all \( s \in [0, t] \), since we assume that \( v_0 \neq 0 \). Hence, the right hand side of equation (22) approaches zero as \( \epsilon \to 0 \).

Therefore \( S_t^e \to S_t^s \) in \( L_2(\Omega) \) as \( \epsilon \to 0 \). This convergence does not depend on \( t \) and is hence uniform with respect to \( t \in [0, T] \). \( \square \)

6. Simulation Examples

Let us consider the Thai stock market. Figure 1 shows the daily prices of the data set consisting of 283 open-prices of the K-BANK between July 2, 2008 and June 30, 2009. The empirical data set for these stock prices were obtained from http://www.set.or.th/. Figure 2 shows the log returns of the stock prices in that period.
Figure 1. Stock prices trading daily of K-BANK between July 2, 2008 and June 30, 2009.

Figure 2. Log returns on the stock prices of K-BANK between July 2, 2008 and June 30, 2009.
The statistics of stock prices and log returns are given in Table 1.

### Table 1. Statistics of K-BANK data set

<table>
<thead>
<tr>
<th></th>
<th>Stock prices</th>
<th>Log returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data amount</td>
<td>238</td>
<td>237</td>
</tr>
<tr>
<td>Mean</td>
<td>55.2153</td>
<td>0.0001</td>
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<tr>
<td>Standard deviation</td>
<td>10.3166</td>
<td>0.0311</td>
</tr>
<tr>
<td>Skewness</td>
<td>0.2856</td>
<td>0.2990</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>1.6902</td>
<td>4.5338</td>
</tr>
</tbody>
</table>

Figure 3 shows the empirical data of K-BANK open-price as compared to the price simulated by the classical gBm with compound Poisson jumps and stochastic volatility. The simulated model is

\[
S_t = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \nu_s ds + \int_0^t \sqrt{\nu_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right],
\]

and

\[
dv_t = (\omega - \theta v_t) dt + \xi v_t d\tilde{W}_t.
\]

The model parameters \(\mu = -3.125\) and \(\sigma = 0.0311\). The mean of jumps = 0.0425, the standard deviation of jumps = 0.0175 and the intensity \(\lambda = 17\). The model parameters for stochastic volatility are \(\omega = 0.00525\), \(\xi = 0.2250\) and \(\theta = 0.000825\). For comparative purposes, we compute the Average Relative Percentage Error (ARPE). By definition,

\[
\text{ARPE} = \frac{1}{N} \sum_{k=1}^N \left| \frac{X_k - Y_k}{X_k} \right| \times 100,
\]

where \(N\) is the number of prices, \(X = \{X_k\}_{k \geq 1}\) is the market price and \(Y = \{Y_k\}_{k \geq 1}\) is the model price. After working 105 trails we compute ARPE for Figure 3 which will be denoted by ARPE (3).
Figure 3. Price behavior of K-BANK, between July 2, 2008 and June 30, 2009, as compared with a scenario simulated from a gBm with a fractional volatility model. (Solid line := empirical data, dashed line := simulated by

\[ S_t = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \sigma^2_s ds + \int_0^t \sqrt{\sigma_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right] \]

with stochastic volatility

\[ dv_t = (\omega - \theta v_t) dt + \xi v_t dW_t, \]

\[ N = 105, \text{ ARPE(3) := 21.23404.} \]

Figure 4 shows the empirical data of K-BANK open-price as compared to the price that was simulated by an approximation of gBm with compound Poisson jumps and fractional stochastic volatility. The simulated model is

\[ S_t^{\text{emp}} = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \sigma^{\text{emp}}_s ds + \int_0^t \sqrt{\sigma^{\text{emp}}_s} dW_s + \int_0^t \log(1 + Y_s) dN_s \right] \]

and

\[ v_t^{\text{emp}} = \left( v_0 + \omega \int_0^t \exp(\kappa s - \xi B_s) ds \right) \exp(\xi B_t - \kappa t). \]
The value of $\mu$, $\sigma$ and the parameter for jumps and volatility are the same as Figure 3. For the remaining data, we choose $\varepsilon = 0.00001$ and $\alpha = 0.00125$.

**Figure 4.** Price behavior of K-BANK, between July 2, 2008 and June 30, 2009, as compared with a scenario simulated from a gBm with a fractional volatility model. (Solid line := empirical data, dashed line := simulated by

$$S^\varepsilon_t = S_0 \exp \left[ \mu t - \frac{1}{2} \int_0^t \nu_s^\varepsilon ds + \int_0^t \sqrt{\nu_s^\varepsilon} dW_s + \int_0^t \log(1 + Y_s) dN_s \right]$$

with stochastic volatility

$$\nu_t^\varepsilon = \left( \nu_0 + \omega \int_0^t \exp(\kappa s - \xi B^B_s) ds \right) \exp(\xi B^B_t - \kappa t),$$

$N = 105$, ARPE := 19.54647,

where ARPE(4) is the ARPE for Figure 4.)

By comparing the ARPE of Figures 3 and 4, we can see that in case of K-BANK, the sample path of an approximation of gBm with compound Poisson jumps and fractional stochastic volatility gives a better fit with the data than the classical gBm with compound Poisson jumps and stochastic volatility.
Figure 5 shows the convergence of ARPE(3) and ARPE(4), working out for 25, 45, 55, 65, 75, 95 and 105 trails.

![Figure 5: Convergence of ARPE(3) and ARPE(4)](image)

Figure 5. Convergence of ARPE(3) and ARPE(4) with $N = 25, 45, 55, 95$ and 105.

References


