Lattice Tree Versus Dynamic Programming in Real Option Analysis

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Abstract : This paper discusses the influences of the Chinese cellulosic ethanol policy based on lattice tree and dynamic programming approaches in real option analysis. Based on the basic hypothesis of double stochastic variables and two construction stages, the empirical conclusions are shown in each real option model respectively. Because of the existence of by-products, the influence of the subsidy is not obviously to the optimal decisions. Thus, improving the technology to fine the high value by-products is the effective way to increase the revenues.

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1 Introduction

Real option analysis usually can be divided into two categories: numerical and analytic methods. Lattice tree is a simple numerical method, such as binomial

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lattice tree can be used to approximate the underlying stochastic process. While, dynamic programming is another general and useful tool for dynamic optimization problem under uncertainty in real option analysis. It can be used to derive the analytic solution for a real option model.

The concept of lattice tree is based on the construction of a tree that starts with an initial value of the state variable. It includes the binomial tree, the ternary tree and other more complex cases. Cross [1] proposed the standard binomial option pricing model, which is known as the Cox-Ross-Rubinstein (CRR) binomial tree. Boyle [2] extended the binomial lattice tree to a trinomial lattice tree option pricing model, which is conceptually similar to the CRR model. The difference is that the trinomial lattice tree model has three jump parameters (up, down and stable or middle path) and three related probabilities. Furthermore, Mandan [3] presented the N-tree model. The assumption of the N-tree model is about the underlying asset pricing options for some discrete stochastic process. The price movement of a tree structure is formed, the value of the option can be extrapolated from the end node of the tree.

Dynamic programming method is based on splitting the decisions in parts that comprise a sequence in time, and it aims to find the optimal path of decisions. It breaks a whole sequence of decisions into two components: the immediate decision and some future decisions. Since there is no decision pending at the last decision point, working backwards can derive the optimal path starting from the initial decision point.The time in dynamic programming method can be considered as either discrete or continuous. Binomial lattice tree is a typical discrete time case of dynamic programming.

The remainder of this paper is organized as follows: Section 2 expounds the basic hypothesis. Section 3 indicates the parameters used in the hypothesis. Section 4 constructs the real option model by lattice tree method, and shows the basic analysis. Section 5 constructs the real option model by dynamic programming approach, and explains the analysis results. Section 6 shows the conclusions.

2   The Basic Hypothesis

Because of the severely energy crisis and environmental issues in recent years, most countries develop renewable energy vigorously. It is well known there are many uncertainties and risks in the renewable energy development, the promotion of the renewable energy such as the cellulosic ethanol project needs the support of government policies. Thus, this paper discusses the real option models based on the investment of cellulosic ethanol plants in China.

The significant difference with other countries is that the project and price of fuel ethanol is determined by the government of China, the fuel ethanol price is set at 0.9111 times the price of No.93 gasoline. Thus the price of gasoline is one key factor in the cellulosic ethanol investment. Assume that the gasoline price is stochastic variable, and the same to the corn cob price considered as the main raw material.

Besides the assumption of double stochastic variables, another hypothesis is that there are two investment construction stages before putting into production in commission in the cellulosic ethanol investment. Specially, each construction stage is irreversible. This paper will use lattice tree and dynamic programming approaches to establish two real option models to investigate the influence of subsidy policy.

3   Parameters Considered in Real Option Model

Based on the report of Kang [4], each ten tons corn cob can produce 1.5 tons ethanol, 1.2 tons xylitol and 1 ton pure lignin. Meanwhile, producing every ton cellulosic ethanol needs 2600 Chinese yuan zymin as another important raw material. At the same time, the producer can obtain the subsidy from the Chinese government. Let symbols P^g, P^c, P^x, P^l and P^z represent the prices of gasoline, corn cob, xylitol, pure lignin and corn cob. Because of the Chinese subsidy policy, let symbol S denote the subsidy for every ton cellulosic ethanol.

Except the gasoline price and corn cob price, the constants values can easily obtained from the government documents, conference reports and the announcements of the first cellulosic ethanol producer with annual capacity of 50,000 tons. Based on the treasury bond in China, we use the average interest of treasury bond in early 2015 to represent the discount rate. Here, symbol Q represents the capacity of the cellulosic ethanol plant, symbol r denotes the discount rate.

By the assumption of two construction stages. Assume that J_1 is the cost of stage-1 construction, J_2 is the cost of stage-2 construction, that is, $J_1 = aC_{other}$, $J_2 = (1 - a)C_{other}$. Here, C_{other} is all the investment construction cost in the cellulosic ethanol project, a is the proportion of the stage-1 construction cost.

All these constant parameters are shown as Table 1.

Table 1: The values of the constant parameters

parameter	value	parameter	value
P^x	23,000 yuan/ton	P^l	4,500 yuan/ton
P^z	2,600 yuan/ton cellulosic ethanol	C_{other}	166,000,000 yuan
S	800 yuan/ton cellulosic ethanol	r	0.032
Q	50,000 tons	a	0.5

According to the basic hypothesis, using the method introduced in Guthrie [5], the decision tree can be shown as Figure 1. Here, the label W stands for action "wait", the label I stands for action "invest".
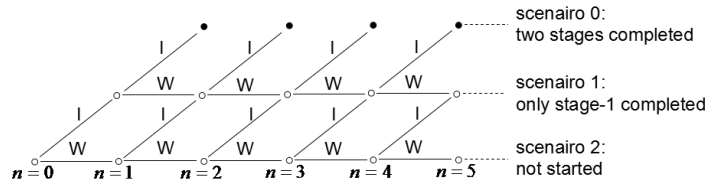
Figure 1: Decision tree for the two construction stages investment

As Figure 1 shown, the investors must make a decision at date 0 that investing immediately or waiting for more information until the next date. If the investors wait, there will be zero cash flow at date 0 and all stages are not started. If the investors decide to construct the first stage, there will be a cash flow of $-J_1$ at date 0 and only the second construction stage is remained. At date 1, the first construction stage will be either completed or not. If it has not been started, the investors face the same situation as date 0. But if the first construction stage has been completed, the investors will face two choices again, investing the second construction stage right now or still waiting for more information. If they wait, there is also zero cash flow at date 1 and only the first construction stage is completed. On the contrary, if they undertake the second construction stage, there will be a capital expenditure of J_2 at date 1 and all these two stages will be completed at date 2. Therefore, the cellulosic ethanol project will be in one of three scenarios starting from date 2: two stages completed, only stage-1 completed and not started, which are denoted as scenario 0, scenario 1 and scenario 2.

4   Real Option Model Based on Lattice Tree

By the assumption, the gasoline price and the corn cob price are stochastic variables, suppose that they follow Geometric Brownian Motions. Let $P^g(i,n)$ denote the price of gasoline with n periods elapsed and i downward movements, $P^c(j,n)$ denote corn cob price with n periods elapsed and j downward movements, where $0 \leq n \leq T$, $0 \leq i \leq n$, $0 \leq j \leq n$. T is the number of time periods. In particular, the gasoline price and the corn cob price at the next period can be presented as follows

$$P^g(i, n+1) = U_g P^g(i,n), \text{ with probability } p_g, \tag{4.1}$$

$$P^g(i+1, n+1) = D_g P^g(i,n), \text{ with probability } 1 - p_g, \tag{4.2}$$

$$P^c(j, n+1) = U_c P^c(j,n), \text{ with probability } p_c, \tag{4.3}$$

$$P^c(j+1, n+1) = D_c P^c(j,n), \text{ with probability } 1 - p_c. \tag{4.4}$$

Here, U_g is the range of gasoline price upward movements. D_g is the range of gasoline price downward movements. p_g is the risk-neutral probability of gasoline price increase. Similarly, U_c is the size of corn cob price upward movements. D_c is

the size of corn cob price downward movements. p_c is the risk-neutral probability of corn cob price increase.

Using the method introduced in the book of Kodukula and Papudesu [6], based on the daily history of No.93 gasoline price of Shandong province between March 1. 2011 and May 31. 2015, the volatility of gasoline price with the year as unit equals 0.12. Furthermore, the range of gasoline price upward movements equals 1.12 and the risk-neutral probability of gasoline price increase equals 0.61. Similarly, based on the daily history data of corn cob price between March 1. 2012 and May 31. 2015, The volatility of corn cob price with the year as unit is 0.77, the range of corn cob price upward movements is 2.17, the risk-neutral probability of corn cob price increase is 0.33. All these parameters values are shown in Table 2.

Table 2: The values of the constant parameters

parameter	description	value	note
$P^g(0,0)$	the initial price of gasoline	8,368 yuan/ton	the average price of No.93 gasoline price in Shandong province from Jan. to May. in 2015
σ_g	the volatility of gasoline price	0.12	calculate with the history data between 2011.March.1 and 2015.May.31
U_g	the range of gasoline price upward movements	1.12	$U_g = e^{\sigma_g}$
D_g	the range of gasoline price downward movements	0.89	$D_g = \frac{1}{U_g} = e^{-\sigma_g}$
p_g	the risk-neutral probability of gasoline price increasing	0.61	$p_g = \frac{e^r - D_g}{U_g - D_g}$
$P^c(0,0)$	the initial price of corn cob	451 yuan/ton	the average price of corn cob in Shandong province from Jan. To May. in 2015
σ_c	the volatility of corn cob price	0.77	calculate with the history data between 2012.March.1 and 2015.May.31
U_c	the range of corn cob price upward movements	2.17	$U_c = e^{\sigma_c}$
D_c	the range of corn cob price downward movements	0.46	$D_c = \frac{1}{U_c} = e^{-\sigma_c}$
p_c	the risk-neutral probability of corn cob price increasing	0.33	$p_c = \frac{e^r - D_c}{U_c - D_c}$

From the Middle and Long Term Program of Renewable Energy Development of China, suppose that the cellulosic ethanol investment right will be lost if the construction program is not completed on or before 2020 which is started from

2015. That is, the number of time periods or the expiration date T equals to 5. Therefore, the binomial lattice trees of gasoline price and corn cob price can be constructed respectively. Using the average prices of No.93 gasoline and corn cob in Shandong province from Jan. to May. in 2015 as the initial data, that is, $P^g(0,0) = 8368$ yuan per ton, $P^c(0,0) = 451$ yuan per ton, the binomial lattice trees of gasoline price and corn cob price can be shown as Table 3 and Table 4.

Table 3: The value in the binomial lattice tree of gasoline price (yuan/ton)

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$i=0$	8368	9372	10497	11756	13167	14747
$i=1$		7448	8341	9342	10463	11719
$i=2$			6628	7424	8315	9312
$i=3$				5899	6607	7400
$i=4$					5250	5880
$i=5$						4673

Note: In this table, the value at node $(i, n+1)$ equals to $U_g = 1.12$ times the value at node (i, n), the value at node $(i+1, n+1)$ equals to $D_g = 0.89$ times the value at node (i, n).

Table 4: The value in the binomial lattice tree of corn cob price (yuan/ton)

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$j=0$	451	979	2124	4608	10000	21701
$j=1$		207	450	977	2120	4600
$j=2$			95	207	449	975
$j=3$				44	95	95
$j=4$					20	44
$j=5$						9

Note: In this table, the value at node $(j, n+1)$ equals to $U_c = 2.17$ times the value at node (j, n), the value at node $(j+1, n+1)$ equals to $D_c = 0.46$ times the value at node (j, n).

Let $X(i, j, n)$ to be the market value of the completed project at node (i, j, n), then

$$X(i,j,n) = Q(0.9111P^g(i,n) + \frac{4}{5}P^x + \frac{2}{3}P^l - \frac{20}{3}P^c(j,n) - P^z + S), \qquad (4.5)$$

Here, $0 \leq n \leq T$, $0 \leq i, j \leq n$.

Since there are two state variables, i.e. variable i represents the gasoline price and variable j represents the corn cob price, then we wish to incorporate multiple state variables. To do this, we generalize the notion of Guthrie [5]. Let $V_h(i, j, n)$ denote the market value of the investment right at date n if there are i downward movements in the first state variable and j downward movements in the second one. Here, $h = 0, 1, 2$ represents the number of stages of construction remaining to be completed.

Case 1: Scenario 0 - two stages completed

If the construction program is completed immediately, the investment right is worth whatever the owners will obtain from the imminent sale value. Therefore, the decision value functions can be written as

$$V_0(i, j, n) = -aC_{other} - e^{-r}(1 - a)C_{other} + X(i, j, n). \qquad (4.6)$$

where $0 \leq n \leq T$, $0 \leq i, j \leq n$. Here, r is the risk-free interest rate.

Case 2: Scenario 1 - only stage-1 completed

Since the investment right will be lost if the construction program is not completed on or before expiration date T. It satisfies terminal conditions as

$$V_1(i, j, T) = 0, \qquad (4.7)$$

where $0 \leq i, j \leq T$.

For each date, both gasoline price and corn cob price can increase or decrease with the related probability. Hence, there are four cases in the next date as Figure 2 shown. This quadrinomial lattice is called bidimensional binomial lattice approach named by Fan [7]. This paper uses the same idea to construct the real option model.
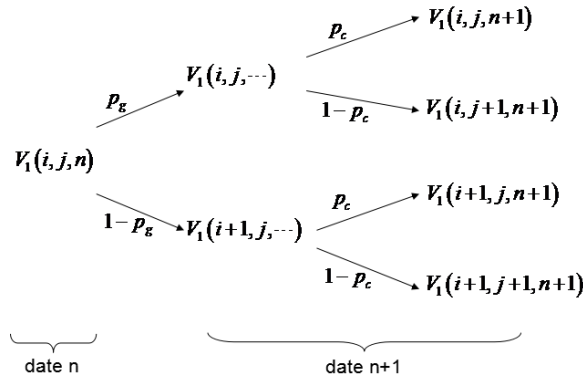

Figure 2: Stochastic decision-making process with two state variables

By backward induction, the lattices for decision value $V_1(i, j, n)$ can be filled by the following formula

$$
\begin{aligned}
V_1(i, j, n) = \max\{ &-J_2 + X(i, j, n), \\
&e^{-r}[p_g[p_c V_1(i, j, n + 1) + (1 - p_c)V_1(i, j + 1, n + 1)] + \\
&(1 - p_g)[p_c V_1(i + 1, j, n + 1) + (1 - p_c)V_1(i + 1, j + 1, n + 1)]]\},
\end{aligned}
$$
$$(4.8)$$

where $0 \leq n \leq T$, $0 \leq i, j \leq n$.

Thus, the government and investors can choose the action that maximized the market value of the project.

Case 3: Scenario 2 - not started

Scenario 2 has the same terminal conditions. Since the construction program is never started, the project right value must equal 0 at expiration date T.

$$V_2(i, j, T) = 0, \tag{4.9}$$

where $0 \leq i, j \leq T$.

Similarly, the last line of lattice trees for $V_2(i, j, n)$ can be filled by the terminal conditions. Then the decision values of the investment right at each node can be calculated by backward induction based on the following equation

$$
\begin{aligned}
V_2(i, j, n) = \max\{&-J_1 + e^{-r}[p_g[p_c V_1(i, j, n+1) + (1 - p_c)V_1(i, j+1, n+1)] + \\
&(1 - p_g)[p_c V_1(i+1, j, n+1) + (1 - p_c)V_1(i+1, j+1, n+1)]], \\
&e^{-r}[p_g[p_c V_2(i, j, n+1) + (1 - p_c)V_2(i, j+1, n+1)] + \\
&(1 - p_g)[p_c V_2(i+1, j, n+1) + (1 - p_c)V_2(i+1, j+1, n+1)]]\},
\end{aligned}
\tag{4.10}
$$

where $0 \leq n \leq T$, $0 \leq i, j \leq n$.

Based on the binomial lattices trees of the gasoline price and corn cob price, followed all these value functions, the decision values of the cellulosic ethanol investment with two stochastic variables and double construction stages at each scenario are shown in Table 5. Here, the number 0 stands for Scenario 0, the number 1 stands for Scenario 1, the number 2 stands for Scenario 2.

Note: In the year 2015+k ($k = 0, 1, 2, 3, 4, 5$), both gasoline and corn cob prices can decrease k times. From top to bottom, the first $k + 1$ rows stand for the gasoline price decreases 0 time, the second $k + 1$ rows stand for the gasoline price decreases 1 time, and so on. In each $k + 1$ rows, the values present the corn cob price decreases from 0 to k times.

Comparing these three scenarios, under investors perspective, the decision value at scenario 0 equals 1047 million yuan, which is less than 1127 million yuan at scenario 1, but it is a little more than 1026 million yuan at scenario 2 in 2015. If the investors have completed the construction immediately, the decision values from 2015 to 2017 are greater than zero as well. With the time elapsed from 2018 to 2020, the decision values are less than zero when the gasoline price decreases 0 time. Clearly, the benefit decreases with the increase of costs. The decision values at Scenario 1 show that it is better to complete the stage-1 construction one year in advance as well. At scenario 2, although the government offers the subsidy for the cellulosic ethanol project, the investors are still better to choose action wait at the last two years.

In addition, the managerial flexibility can be well underestimated using real option analysis. The values in boldface at Scenario 0 present that the investors will suffer loss if the construction is completed at these nodes. The values with underline at Scenario 1, and the values with overline at Scenario 2 indicate that choosing action wait is better at these nodes.

Table 5: The value in the binomial lattice tree of corn cob price (yuan/ton)

2015			2016			2017			2018			2019			2020		
0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
1047	1127	1026	917	1026	943	586	835	752	**-183**	481	398	**-1917**	0	$\bar{0}$	**-5745**	0	0
			829	941	858	488	747	664	**-293**	408	325	**-2040**	0	$\bar{0}$	**-5883**	0	0
			1174	1254	1143	410	678	595	**-381**	349	266	**-2138**	0	$\bar{0}$	**-5992**	0	0
			1086	1167	1055	1144	1225	1114	**-450**	303	220	**-2215**	0	$\bar{0}$	**-6079**	0	0
						1046	1126	1016	1026	1106	997	**-2277**	0	$\bar{0}$	**-6149**	0	0
						968	1048	938	916	996	887	709	790	$\bar{0}$	**-6204**	0	0
						1262	1343	1231	829	909	800	586	667	$\bar{0}$	**-44**	0	0
						1164	1245	1133	759	840	730	488	569	$\bar{0}$	**-182**	0	0
						1086	1167	1055	1283	1363	1251	410	491	$\bar{0}$	**-292**	0	0
									1173	1253	1141	349	429	$\bar{0}$	**-379**	0	0
									1085	1166	1054	1266	1347	$\bar{0}$	**-448**	0	0
									747	1096	985	1143	1223	$\bar{0}$	**-503**	0	0
									1337	1417	1305	1045	1125	$\bar{0}$	1163	0	0
									1227	1307	1195	967	1048	$\bar{0}$	1025	0	0
									1140	1220	1108	905	986	$\bar{0}$	915	0	0
									1070	1151	1039	1384	1465	$\bar{0}$	828	0	0
												1261	1341	$\bar{0}$	759	0	0
												1163	1244	$\bar{0}$	704	0	0
												1085	1166	$\bar{0}$	1419	0	0
												1024	1104	$\bar{0}$	1281	0	0
												1409	1490	$\bar{0}$	1171	0	0
												1286	1366	$\bar{0}$	1084	0	0
												1188	1269	$\bar{0}$	1015	0	0
												1110	1191	$\bar{0}$	960	0	0
												1049	1129	$\bar{0}$	1473	0	0
															1335	0	0
															1226	0	0
															1139	0	0
															1069	0	0
															1014	0	0
															1485	0	0
															1347	0	0
															1237	0	0
															1150	0	0
															1081	0	0
															1026	0	0

5　Real Option Model Based on Dynamic Programming

In the infinite time horizon case, assume that the prices of unit revenue P and unit expenditure C follow Geometric Brownian Motions, $dP = \mu_P P dt + \sigma_P P dB_{1t}$, $dC = \mu_C C dt + \sigma_C C dB_{2t}$. Here, μ_P is the drift of the unit revenue P, σ_P is the volatility of the unit revenue P. μ_C is the drift of the unit expenditure C, σ_C is the volatility of the unit expenditure C. Furthermore, the Brownian motions B_{1t} and B_{2t} satisfy that $dB_{1t}dB_{2t} = \rho dt$, ρ is the correlation coefficient of these two Brownian Motions.

Step 1: calculate the value of the completed project

Let $V(P, C)$ denote the value of the completed cellulosic ethanol project, then it can be determined by the dynamic programming method in real option analysis. According to the Bellman equation

$$rVdt = \pi dt + E_t[dV], \tag{5.1}$$

where r is the discount rate, $E_t[\cdots]$ is the conditional expectation at the current time t, $\pi(P,C) = max\{P - C, 0\}$ is the instantaneous profit cash flow.

From the Itos lemma, the Bellman equation can be expressed as a partial differential equation as follows

$$\frac{1}{2}\sigma_P^2 P^2 V_{PP} + \frac{1}{2}\sigma_C^2 C^2 V_{CC} + \rho\sigma_P\sigma_C V_{PC} + \mu_P P V_P + \mu_C C V_C - rV + \pi = 0, \quad (5.2)$$

with boundary conditions

$$V(0,C) = 0, \tag{5.3}$$

$$V(P,\infty) = 0, \tag{5.4}$$

$$V(P,C) \text{ is continuous at } P = C, \tag{5.5}$$

$$V_P(P,C), V_C(P,C) \text{ are continuous at } P = C. \tag{5.6}$$

In this case of $P < C$, $\pi(P,C) = 0$, so the equation (5.2) will be

$$\frac{1}{2}\sigma_P^2 P^2 V_{PP} + \frac{1}{2}\sigma_C^2 C^2 V_{CC} + \rho\sigma_P\sigma_C V_{PC} + \mu_P P V_P + \mu_C C V_C - rV = 0. \quad (5.7)$$

Using the same idea in Dixit and Pindyck [8] to solve the equation (5.7), this kind of partial differential equation can be solved by reducing to a one variable problem based on the homogeneity of the value function. Let m denote the ratio value $\frac{P}{C}$, then the optimal decision should only depend on the ratio m (In the case $P < C$, the ratio $m = \frac{P}{C} < 1$). Correspondingly, the value function should be homogeneous of degree 1 in (P,C), that is, the value function can be written as $V(P,C) = Cf(m)$. Here, function $f(m)$ is unknown.

By the derivation rules, the equation (5.7) can be changed as an ordinary differential equation as follows

$$\frac{1}{2}(\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2)m^2 f''(m) + (\mu_P - \mu_C)mf'(m) + (\mu_C - r)f(m) = 0. \quad (5.8)$$

The characteristic equation of this ordinary differential equation is

$$\frac{1}{2}(\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2)\beta(\beta - 1) + (\mu_P - \mu_C)\beta + (\mu_C - r) = 0. \tag{5.9}$$

The roots are

$$\beta_1 = (\tfrac{1}{2} - \tfrac{\mu_P - \mu_c}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2}) + \sqrt{(\tfrac{1}{2} - \tfrac{\mu_P - \mu_c}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2})^2 - \tfrac{2(\mu_C - r)}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2}}, \quad (5.10)$$

$$\beta_2 = (\tfrac{1}{2} - \tfrac{\mu_P - \mu_c}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2}) - \sqrt{(\tfrac{1}{2} - \tfrac{\mu_P - \mu_c}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2})^2 - \tfrac{2(\mu_C - r)}{\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2}}, \quad (5.11)$$

Suppose that $r > \mu_P > \mu_C$, then $\beta_1 > 1$, $\beta_2 < 0$.

According to the homogeneity and the boundary condition (5.3), the general solution of equation (5.9) is just a linear combination of the two power solutions

corresponding to these two roots, that is, $f(m) = Am^{\beta_1} + Bm^{\beta_2}$, $m < 1$, $\beta_1 > 1$, $\beta_2 < 0$. Furthermore, by the boundary conditions (5.3) and (5.4), the function $f(m)$ satisfies that $f(0) = 0$, $f(m) \to 0$ as $m \to 0$. This requires $B = 0$, so that $f(m) = Am^{\beta_1}$, $m < 1$, $\beta_1 > 1$. Thus,

$$V(P,C) = AP^{\beta_1}C^{1-\beta_1}, P < C, \beta_1 > 1. \tag{5.12}$$

In this case of $P \geq C$, the equation (5.2) will be

$$\frac{1}{2}\sigma_P^2 P^2 V_{PP} + \frac{1}{2}\sigma_C^2 C^2 V_{CC} + \rho\sigma_P\sigma_C V_{PC} + \mu_P P V_P + \mu_C C V_C - rV + P - C = 0. \tag{5.13}$$

Using the same method, the partial differential equation (5.13) will be changed as a new ordinary differential equation as follows

$$\frac{1}{2}(\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2)m^2 f''(m) + (\mu_P - \mu_C)mf'(m) + (\mu_C - r)f(m) + m - 1 = 0. \tag{5.14}$$

Here, the ratio $m \geq 1$. Since the equation (5.14) has the same characteristic equation as equation (5.9), the roots β_1, β_2 are the same as equations (5.10) and (5.11). Thus, the general solution of the inhomogeneous ordinary equation has two parts: one is the combination of the power solutions of the homogeneous part, another is one particular solution of the inhomogeneous equation. In fact, the function $f(m)$ also satisfies $f(m) \to 0$ as $m \to \infty$. This shows that the combination of the power solutions of the homogeneous part will be $f(m) = Em^{\beta_2}$, $\beta_2 < 0$. Since $f(m) = \frac{m}{r-\mu_P} - \frac{1}{r-\mu_C}$ is one of the particular solution of equation (5.14). Thus the general solution can be written as $f(m) = Em^{\beta_2} + \frac{m}{r-\mu_P} - \frac{1}{r-\mu_C}$, $m \geq 1$, $\beta_2 < 0$. That is,

$$V(P,C) = EP^{\beta_2}C^{1-\beta_2} + \frac{P}{r - \mu_P} - \frac{C}{r - \mu_C}, P \geq C, \beta_2 < 0. \tag{5.15}$$

Based on these two cases, the value function of the completed project is

$$V(P,C) = \begin{cases} AP^{\beta_1}C^{1-\beta_1} & P < C, \beta_1 > 1, \\ EP^{\beta_2}C^{1-\beta_2} + \frac{P}{r-\mu_P} - \frac{C}{r-\mu_C}, & P \geq C, \beta_2 < 0. \end{cases} \tag{5.16}$$

where $\beta_1 > 1$, $\beta_2 < 0$.

According to the boundary conditions (5.5) and (5.6), the coefficient parameters satisfy

$$A = \frac{1}{\beta_1 - \beta_2}\left(\frac{1 - \beta_2}{r - \mu_P} - \frac{\beta_2}{r - \mu_C}\right) > 0, \tag{5.17}$$

$$E = \frac{1}{\beta_1 - \beta_2}\left(\frac{1 - \beta_1}{r - \mu_P} - \frac{\beta_1}{r - \mu_C}\right) < 0. \tag{5.18}$$

Step 2: calculate the option value of Scenario 0

At scenario 0, all the two construction stages are completed, the option value is about the difference of the values of completed project and construction cost, so the option value at this scenario $F0(P, C)$ satisfies the following Bellman equation

$$rF0dt = r(V - \frac{J_2}{Q} - \frac{J_1}{Q})dt. \tag{5.19}$$

In fact, it is easily to get the analytic form of the option value function $F0(P, C)$.

$$F0(P, C) = \begin{cases} AP^{\beta_1}C^{1-\beta_1} - \frac{J_2}{Q} - \frac{J_1}{Q} & \frac{P}{C} < 1, \beta_1 > 1, \\ EP^{\beta_2}C^{1-\beta_2} + \frac{P}{r-\mu_P} - \frac{C}{r-\mu_C} - \frac{J_2}{Q} - \frac{J_1}{Q}, & \frac{P}{C} \geq 1, \beta_2 < 0. \end{cases} \tag{5.20}$$

Step 3: calculate the option value of Scenario 1.
If only the stage-1 construction has been completed, the owners will face two choices: invest or wait. If the owners choose to invest, they must pay the investment cost of stage-2 construction, or else they still have only completed the stage-1 construction. Thus the Bellman equation of the option value $F1(P, C)$ is

$$rF1dt = max\{r(V - \frac{J_2}{Q})dt, E_t[dF1]\}. \tag{5.21}$$

In the wait or continue region, $rF1dt = E_t[dF1]$, we can get a partial differential equation (5.22) by the Itos lemma and conditional expectation

$$\frac{1}{2}\sigma_P^2 P^2 F1_{PP} + \frac{1}{2}\sigma_C^2 C^2 F1_{CC} + \rho\sigma_P\sigma_C F1_{PC} + \mu_P P F1_P + \mu_C C F1_C - rF1 = 0, \tag{5.22}$$

with boundary conditions

$$F1(0, C) = 0, \tag{5.23}$$

$$F1(P, \infty) = 0, \tag{5.24}$$

$$F1(P_1^*, C_1^*) = V(P_1^*, C_1^*) - \frac{J_2}{Q}, \tag{5.25}$$

$$F1_P(P_1^*, C_1^*) = V_P(P_1^*, C_1^*), \tag{5.26}$$

$$F1_C(P_1^*, C_1^*) = V_C(P_1^*, C_1^*). \tag{5.27}$$

P_1^*, C_1^* are the critical values. Equation (5.25) is called "value-matching condition", equations (5.26) and (5.27) are called "smooth-pasting condition".

Using the same method, let the ratio value $m = \frac{P}{C}$, then the option value function $F1(P, C)$ can be written as $F1(P, C) = Cf_1(m)$, so the equation (5.22) will be changed to an ordinary differential equation

$$\frac{1}{2}(\sigma_P^2 - 2\rho\sigma_P\sigma_C + \sigma_C^2)m^2 f_1''(m) + (\mu_P - \mu_C)m f_1'(m) + (\mu_C - r)f_1(m) = 0. \tag{5.28}$$

Similarly, the solution has the form $f_1(m) = A_1 m^{\beta_1} + B_1 m^{\beta_2}$, here β_1, β_2 are the same as equation (5.10) and equation (5.11). By the boundary conditions (5.23)

and (5.24), the function $f_1(m)$ satisfies $f_1(m) \to 0$, as $m \to \infty$, this requires that $B_1 = 0$. Thus, the solution is $f_1(m) = A_1 m^{\beta_1}$, $\beta_1 > 1$. That is,

$$F1(P,C) = A_1 P^{\beta_1} C^{1-\beta_1}, \beta_1 > 1. \tag{5.29}$$

According to the value-matching condition and the smooth-pasting conditions, let $m_1^* = \frac{P_1^*}{C_1^*}$ is the ratio of the critical value, then m_1^* is the root of equation

$$E(\beta_1 - \beta_2)m^{\beta_1} + (\beta_1 - 1)\frac{m}{r - \mu_P} - \beta_1 \frac{1}{r - \mu_C} = 0, \tag{5.30}$$

and $m_1^* \geq 1$. Furthermore,

$$A_1 = E\frac{\beta_2}{\beta_1}m_1^{*\beta_2 - \beta_1} + \frac{1}{\beta_1}\frac{1}{r - \mu_P}m_1^{*1-\beta_1}, \tag{5.31}$$

$$C_1^* = \frac{\frac{J_2}{Q}}{-A_1 m_1^{*\beta_1} + Em_1^{*\beta_2} + \frac{m_1^*}{r-\mu_P} - \frac{1}{r-\mu_C}}, \tag{5.32}$$

$$P_1^* = \frac{m_1^* \frac{J_2}{Q}}{-A_1 m_1^{*\beta_1} + Em_1^{*\beta_2} + \frac{m_1^*}{r-\mu_P} - \frac{1}{r-\mu_C}}. \tag{5.33}$$

Hence,

$$F1(P,C) = \begin{cases} A_1 P^{\beta_1} C^{1-\beta_1}, & \frac{P}{C} < \frac{P_1^*}{C_1^*}, \beta_1 > 1, \\ EP^{\beta_2} C^{1-\beta_2} + \frac{P}{r-\mu_P} - \frac{C}{r-\mu_C} - \frac{J_2}{Q}, & \frac{P}{C} \geq \frac{P_1^*}{C_1^*} \geq 1, \beta_2 < 0. \end{cases} \tag{5.34}$$

    Step 4: calculate the option value of Scenario 2

    If the investors start the project, the owners also have two choices: invest the stage-1 construction right now or still wait. So the option value function $F2(P,C)$ satisfies the Bellman equation

$$rF2dt = max\{r(F1 - \frac{J_1}{Q})dt, E_t[dF2]\}. \tag{5.35}$$

Likewise, $rF2dt = E_t[dF2]$ presents the "wait" or "continue" region. According to the partial differential equation

$$\frac{1}{2}\sigma_P^2 P^2 F2_{PP} + \frac{1}{2}\sigma_C^2 C^2 F2_{CC} + \rho\sigma_P\sigma_C F2_{PC} + \mu_P P F2_P + \mu_C C F2_C - rF2 = 0, \tag{5.36}$$

with boundary conditions

$$F2(0,C) = 0, \tag{5.37}$$

$$F2(P,\infty) = 0, \tag{5.38}$$

$$F2(P_2^*, C_2^*) = F1(P_2^*, C_2^*) - \frac{J_1}{Q}, \tag{5.39}$$

$$F2_P(P_2^*, C_2^*) = F1_P(P_2^*, C_2^*), \tag{5.40}$$

$$F2_C(P_2^*, C_2^*) = F1_C(P_2^*, C_2^*). \tag{5.41}$$

the general solution is

$$F2(P,C) = \begin{cases} A_2 P^{\beta_1} C^{1-\beta_1} & \frac{P}{C} < \frac{P_2^*}{C_2^*}, \beta_1 > 1, \\ E P^{\beta_2} C^{1-\beta_2} + \frac{P}{r-\mu_P} - \frac{C}{r-\mu_C} - \frac{J_2}{Q} - \frac{J_1}{Q}, & \frac{P}{C} \geq \frac{P_2^*}{C_2^*} \geq 1, \beta_2 < 0. \end{cases} \tag{5.42}$$

where

$$A_2 = E\frac{\beta_2}{\beta_1} m_2^{*\beta_2-\beta_1} + \frac{1}{\beta_1}\frac{1}{r-\mu_P} m_2^{*1-\beta_1}, \tag{5.43}$$

$$C_2^* = \frac{\frac{J_2+J_1}{Q}}{-A_2 m_2^{*\beta_1} + E m_2^{*\beta_2} + \frac{m_2^*}{r-\mu_P} - \frac{1}{r-\mu_C}}, \tag{5.44}$$

$$P_2^* = \frac{m_2^* \frac{J_2+J_1}{Q}}{-A_2 m_2^{*\beta_1} + E m_2^{*\beta_2} + \frac{m_2^*}{r-\mu_P} - \frac{1}{r-\mu_C}}. \tag{5.45}$$

Here, the threshold ratio value m_2^* is the root of equation (5.30) as well, and $m_2^* = \frac{P_2^*}{C_2^*} \geq m_1^* \geq 1$.

Following the method introduced in Guthrie [5], the parameters μ_P, σ_P can be estimated by the process $\Delta p_j = p_{j+1} - p_j = v_1 + \epsilon_{1,j+1}$, $\epsilon_{1,j+1} \sim N(0, \phi_1^2)$. Here, $p_j = lnP_j$. v_1 is the sample mean of the array , ϕ_1 is the sample standard deviation of array Δp_j. The estimators of μ_P, σ_P can be calculated by $\hat{\mu}_P = \frac{\hat{v}_1}{\Delta t} + \frac{1}{2}\hat{\sigma}_P^2$, $\hat{\sigma}_P = \frac{\hat{\phi}_1}{\sqrt{\Delta t}}$, here $\hat{v}_1$ and $\hat{\phi}_1$ are the estimators of v_1 and ϕ_1.

Similarly, the parameters μ_C, σ_C can be estimated by $\hat{\mu}_C = \frac{\hat{v}_2}{\Delta t} + \frac{1}{2}\hat{\sigma}_C^2$, $\hat{\sigma}_C = \frac{\hat{\phi}_2}{\sqrt{\Delta t}}$. Here, $\hat{v}_2$ and $\hat{\phi}_2$ are the estimators of v_2 and ϕ_2 that in the process $\Delta c_j = c_{j+1} - c_j = v_2 + \epsilon_{2,j+1}$, $\epsilon_{2,j+1} \sim N(0, \phi_2^2)$. Here, $c_j = lnC_j$.

By the definition of the correlation coefficient $\hat{\rho}_{PC} = \frac{Cov(dlnP,dlnC)}{\sqrt{Var(dlnP)Var(dlnC)}}$, the estimator $\hat{\rho}_{PC}$ is the correlation coefficient of array Δp and array Δc, then ρ the estimator of parameter satisfies $\hat{\rho} = \hat{\rho}_{PC}$.

Besides the gasoline price P^g, the xylitol price P^x and the pure lignin price P^l, the subsidy S can be considered as a part of the unit revenue P under investors perspective. Only the corn cob price P^c and the zymin price P^z are considered as the parts of the unit expenditure C. Thus, $P = 0.9111 P^g + \frac{4}{5}P^x + \frac{2}{3}P^l + S$, $C = \frac{20}{3}P^c + P^z$. Using the same data, these parameters estimators under investors perspective are $\hat{\mu}_P = -0.01161337$, $\hat{\sigma}_P = 0.035737192$, $\hat{\mu}_C = -0.0074469005$, $\hat{\sigma}_C = 0.0304436916$, $\hat{\rho}_{PC} = -0.003539494$.

Based on these estimated values, these option value functions can be determined easily. The parameters used in the option value functions are shown as Table 6.

Following Table 6, no matter which stage is constructed by the investors, the critical ratio values m_1^* and m_2^* are the same, but the related critical values C_1^*,

P_1^*, C_2^*, P_2^* are different. Based on the critical ratio value, the investors can choose the action invest when the ratio value $\frac{P}{C}$ is greater than 2.2548. Obviously, each figure of the related option value function in Figure 3 seems similarly to others.

Table 6: The values of parameters used in functions

parameter	value	parameter	value	parameter	value
β_1	1.3458	m_2^*	2.2548	m_1^*	2.2548
β_1	-1.6826	A_2	13.5859	A_1	13.5859
A	25.5293	C_2^*	1.8690e+17	C_1^*	3.7380e+17
E	-6.7918	P_2^*	4.2142e+17	P_1^*	8.4285e+17

Based on real option model with multistage and double stochastic variables for cellulosic ethanol investment by the dynamic programming approach, the critical ratio values are the same for every construction stage, but the critical values are different at Scenario 1 and Scenario 2. Because of the existence of the by-products in the cellulosic ethanol project, the influence of the subsidy seems not very obviously.
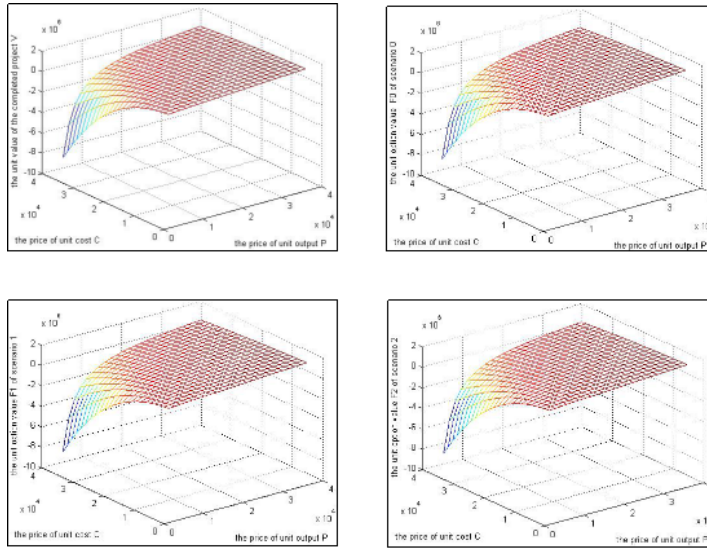

Figure 3: The figures of functions

6   Conclusions

The multistage evaluation model with double stochastic variables based on lattice tree method is a discrete type. Differently, the time horizon in lattice tree model is usually finite. Comparing with these two models, if we only consider two cases - up and down - about the movements of the unit revenue and the

unit expenditure, the option values at each scenario can be obtained following the related option value functions, and these results can be shown as quadrinomial lattice tree. Then the investors can make the optimal decisions of the project. Although the lattice tree method is easily understandable and implemented, there are more uncertainty about the stochastic variables in real life, so it is difficult to handle the complexity as the tree is expanding exponentially as the stochastic factors increase.

However, the time in dynamic programming method can be considered as either discrete or continuous. Based on the infinite time horizon case, the continuous option value functions in dynamic programming model will help the investors to make decisions at any time no matter how the unit revenue and the unit expenditure change. The empirical results indicate that the dynamic programming model is better to predict past expansionary behavior. In addition, the multistage construction and double stochastic variables model is necessary to evaluate the incidence of policy effects relative to revenues and costs.

References

[1] J.C. Cross, S.A. Ross, M. Rubinstein, Option pricing: a simplified approach, Journal of Financial Economics 7 (1979) 229-263.

[2] P. Boyle, Option valuation using a three-jump process, International Option Journal 3 (1986) 7-12.

[3] L.P. Mandan, F. Milne, H. Shefrin, The multinomial option pricing model and its Brownian and Poisson limits, The Review of Financial Studies 2 (1989) 251-265.

[4] D.B. Kang, Biofuel experience in China - Government and Market Development Updates, In: Proceeding of the 6th stakeholder Plenary Meeting of EBTP, 2014.

[5] G. Guthrie, Real Options in Theory and Practice, Oxford University Press, New York, 2009

[6] P. Kodukula, C. Papudesu, Project Valuation: Using Real Options - A Practitioner's Guide, J. Ross Publishing, Florida, 2006.

[7] Y. Fan, J.L. Mo, L. Zhu, Evaluating coal bed methane investment in China based on a real options model, Resources Policy 38 (2013) 50-59.

[8] A.K. Dixit, R.S. Pindyck, Investment under Uncertainty. Princeton University Press, Princeton, New Jersey, 1994.