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Option Pricing under a Mean Reverting Process with Jump-Diffusion and Jump Stochastic Volatility

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Abstract : An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion and exhibits mean reversion. The stochastic volatility follows the jump-diffusion with mean reversion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords : jump-diffusion model; stochastic volatility; characteristic function; option pricing; mean reverting.
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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. All processes that we shall consider in this section will be defined in this space. An asset price model with stochastic volatility has been defined by Heston [1] which has the following dynamics:

$$dS_t = S_t(\mu dt + \sqrt{v_t} dW_t^S), \tag{1.1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v, \tag{1.2}$$

where S_t is the asset price, $\mu \in \Re$ is the rate of return of the asset price, v_t is the volatility of asset returns, $\kappa > 0$ is the rate at which the volatility reverts

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toward its long-term mean, $\theta \in \Re$ is the mean long-term volatility, $\sigma > 0$ is the volatility of the volatility process, W_t^S and W_t^v are standard Brownian motions corresponding to the processes S_t and v_t, respectively, with constant correlation ρ. Bate [2] introduced the jump-diffusion stochastic volatility model by adding log normal jump Y_t to the Heston stochastic volatility model. In the original formulation of Bate, the model has the following form:

$$dS_t = S_t(\mu dt + \sqrt{v_t}dW_t^S) + S_{t-}Y_t dN_t^S, \tag{1.3}$$

where N_t^S is the Poisson process which corresponds to the underlying asset S_t, Y_t is a proportion of jump size of the asset price (1.1) with log normal distribution and S_t- means that there is a jump in the value of the process before the jump is used on the left-hand side of the formula. Eraker et al. [3] extended Bate's work by incorporating jumps into the volatility model, i.e.

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^v + Z_t dN_t^v \tag{1.4}$$

Eraker et al. [3] developed a likelihood-based estimation strategy and provided estimates of parameters, spot volatility, jump times, and jump sizes using S&P 500 and Nasdaq 100 index returns. Moreover, they examined the volatility structure of the S&P and Nasdaq indices and indicated that models with jumps in volatility are preferred over those without jumps in volatility. But they did not provide a closed-form formula for the price of a European call option.

Empirical evidence on mean reversion in financial assets has been produced by Cecchetti et al. [4] and Bessembinder et al. [5], respectively. It has been documented that currency exchange rates also exhibit mean reversion. Jorion and Sweeney [6] show how the real exchange rates revert to their mean levels and Sweeney [7] provides empirical evidence of mean reversion in G-10 nominal exchange rates. Mean reversion also appears in some stock prices as evidenced by Poterba and Summers [8].

In this paper, we consider the problem of finding a closed-form formula for a European call option where the asset price follows mean reverting jump-diffusion and the stochastic volatility with jump.

The rest of this paper is organized as follows. In Section 2, we briefly discuss model descriptions for option pricing. Deriving a formula for a characteristic function is presented in Section 3. Finally, a closed-form formula for a European call option in terms of characteristic functions is presented.

2 Model Descriptions

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists. The asset price S_t under this measure follows a mean reverting jump-diffusion process, and the volatility v_t follows mean reverting with jump, i.e. our models are governed by the following dynamics:

$$dS_t = b\left(a - \ln S_t - \frac{\lambda^S m}{b}\right) S_t dt + \sqrt{v_t} S_t dW_t^S + S_{t-} Y_t dN_t^S \tag{2.1}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v + Z_t dN_t^v \tag{2.2}$$

where $S_t, v_t, \kappa, \theta, \sigma, W_t^S$ and W_t^v are defined as above, $a \in \Re$ is the mean of long-term asset price return, $b > 0$ is the rate at which the asset price return reverts toward its long-term mean, N_t^S and N_t^v are independent Poisson processes with constant intensities λ^S and λ^v respectively. Y_t and Z_t are proportional jump sizes of the asset price (2.1) and the jump size of the volatility process (2.2) respectively. Suppose that Y_t and Z_t are independent and identically distributed sequences with densities $\phi_{Y_t}(y) := \phi_Y(y)$, $\phi_{Z_t}(z) := \phi_Z(z)$ and $EY_t := m$. Moreover, we assume that the jump processes N_t^S and N_t^v are independent of standard Brownian motions W_t^S and W_t^v.

Assume that the asset price S_t and the volatility v_t satisfy equations (2.1) and (2.2) respectively. Let $L_t = \ln S_t$, by the jump-diffusion chain rule, $\ln S_t$ satisfies the SDE

$$dL_t = b\left(a - L_t - \frac{\lambda^S m}{b} - \frac{v_t}{2b}\right) dt + \sqrt{v_t} dW_t^S + \ln(1 + Y_t) dN_t^S. \tag{2.3}$$

3 Characteristic Functions

We denote the characteristic function for $L_T = \ln S_T$ as

$$f(x : t, l, v) = E_{\mathcal{M}}[e^{ixL_T} | L_t = l, v_t = v] \tag{3.1}$$

where $0 \le t \le T$ and $i = \sqrt{-1}$. Here L_t is the mean reverting asset price process with jumps specified by (2.3) and v_t is the volatility process specified by (2.2). The generalized Feynman-Kac theorem [9] implies that $f(x : t, l, v)$ solves the following partial integro-differential equation (PIDE):

$$\begin{aligned} 0 = {} & \frac{\partial f}{\partial t} + b\left(a - l - \frac{\lambda^S m}{b} - \frac{v}{2b}\right)\frac{\partial f}{\partial l} \\ & + \kappa(\theta - v)\frac{\partial f}{\partial v} + \frac{1}{2}v\frac{\partial^2 f}{\partial l^2} + \rho\sigma v\frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f}{\partial v^2} \\ & + \lambda^S \int_{\Re} [f(x : t, l + y, v) - f(x : t, l, v)]\phi_Y(y)dy \\ & + \lambda^v \int_{\Re} [f(x : t, l, v + z) - f(x : t, l, v)]\phi_Z(z)dz. \end{aligned} \tag{3.2}$$

Lemma 3.1. *Suppose that L_t follows the dynamics in (2.3). Then the characteristic function for L_T can be written in the form*

$$f(x:t,l,v) = \exp[B(t,T) + C(t,T)l + D(t,T)v + ixl], \tag{3.3}$$

where

$$\begin{aligned}
B(t,T) &= (\frac{\lambda^S m}{b} - a)ix(e^{-b(T-t)} - 1) - \theta\kappa \int_t^T D(s,T)ds \\
&\quad + (T-t)\lambda^S \int_{\Re} \left[e^{ixy} - 1\right]\phi_Y(y)dy \\
&\quad + (T-t)\lambda^v \int_{\Re} \left[e^{zD(t,T)} - 1\right]\phi_Z(z)dz, \\
C(t,T) &= ix(e^{-b(T-t)} - 1), \\
D(t,T) &= U(e^{-b(T-t)}) + \frac{e^{-\kappa(T-t)}V(e^{-b(T-t)})}{-\frac{1}{U(1)} + \frac{\sigma^2}{2b}\int_1^{e^{-b(T-t)}} h^{\frac{\kappa}{b}-1}V(h)dh},
\end{aligned}$$

$$\begin{aligned}
U(h) &= \frac{2bh}{\sigma^2}\frac{(\sqrt{1-\rho^2} - \rho i)\frac{\sigma x}{2b}\Phi(a^*, b^*, \frac{h}{\zeta}) + \frac{a^*}{b^*\zeta}\Phi(a^*+1, b^*+1, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{h}{\zeta})}, \\
V(h) &= \frac{\Phi^2(a^*, b^*, \frac{1}{\zeta})e^{(\sqrt{1-\rho^2})\frac{\sigma x}{b}(1-h)}}{\Phi^2(a^*, b^*, \frac{h}{\zeta})}, \\
h &= e^{-b(T-t)}, \\
a^* &= \frac{\frac{b^*}{2}(\sqrt{\rho^2-1} + \rho) + \frac{\sigma}{4b}}{\sqrt{\rho^2-1}}, \\
b^* &= 1 - \frac{\kappa}{b}, \\
\zeta &= \frac{-b}{\sigma x\sqrt{1-\rho^2}},
\end{aligned}$$

and $\Phi(\cdot,\cdot,\cdot)$ is the degenerated hypergeometric function.

Proof. From (3.1), it is clear that

$$f(x:T,l,v) = e^{ixl} \tag{3.4}$$

which is the boundary condition of PIDE (3.2). This implies that

$$B(T,T) = C(T,T) = D(T,T) = 0. \tag{3.5}$$

Substituting (3.3) in (3.2) and using the fact that the function f is never zero, we obtain

$$
\begin{aligned}
0 = [&B_t + (ba - \lambda^S m)(C + ix) + \kappa\theta D \\
&+ \lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy + \lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz] \\
&+ [C_t - b(C + ix)]l \\
&+ [D_t + \frac{1}{2}(C + ix) + \frac{1}{2}(C + ix)^2 - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma(C + ix)D]v
\end{aligned} \tag{3.6}
$$

where B_t, C_t and D_t are the partial derivatives with respect to t of functions B, C and D respectively.

This reduces the problem to one of solving three, much simpler, ordinary differential equations:

$$
\begin{aligned}
&B_t + (ba - \lambda^S m)(C + ix) + \kappa\theta D + \lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy \\
&\quad + \lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz = 0
\end{aligned} \tag{3.7}
$$

$$
C_t - b(C + ix) = 0 \tag{3.8}
$$

$$
D_t + \frac{1}{2}(C + ix)(C + ix - 1) - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma(C + ix)D = 0 \tag{3.9}
$$

subject to boundary conditions (3.5).

The solution to equation (3.8) with the boundary condition $C(T, T) = 0$ is given by

$$
C(t, T) = ix(e^{-b(T-t)} - 1). \tag{3.10}
$$

We now consider equation (3.9). Substituting (3.10) in (3.9), one gets

$$
D_t + \frac{1}{2}\left[ixe^{-b(T-t)}\right]\left[ixe^{-b(T-t)} - 1\right] - \kappa D + \frac{1}{2}\sigma^2 D^2 + \rho\sigma ixDe^{-b(T-t)} = 0.
$$

Hence,

$$
D_t = -\frac{1}{2}\sigma^2 D^2 + \left[\kappa - \rho\sigma ixe^{-b(T-t)}\right] D + \frac{1}{2}\left[x^2 e^{-2b(T-t)} + ixe^{-b(T-t)}\right]. \tag{3.11}
$$

Let $h = e^{-b(T-t)}$ and we define a new function $\hat{D}(h(t), T) := D(t, T)$. Then

$$
\begin{aligned}
\frac{\partial D(t, T)}{\partial t} &= \frac{\partial \hat{D}(h, T)}{\partial h}\frac{\partial h}{\partial t} \\
&= be^{-b(T-t)}\frac{\partial \hat{D}(h, T)}{\partial h}.
\end{aligned} \tag{3.12}
$$

Substituting (3.12) into (3.11), we obtain the following Riccati equation

$$\frac{\partial \hat{D}}{\partial h} = -\frac{1}{2bh}\sigma^2 \hat{D}^2 + \left(\frac{\kappa}{bh} - \frac{\rho\sigma i x}{b}\right)\hat{D} + \frac{1}{2b}\left(x^2 h + ix\right). \tag{3.13}$$

We shall solve the second order ODE (3.13) together with the initial condition $\hat{D}(1,T) = 0$. Let

$$\hat{D}(h,T) = \frac{2bhw'(h)}{\sigma^2 w(h)} \tag{3.14}$$

and taking the derivative of (3.14) with respect to h, one gets

$$\begin{aligned} \frac{\partial \hat{D}}{\partial h} &= \left[\sigma^2 w(h)\frac{\partial}{\partial h}(2bhw'(h)) - 2bhw'(h)\frac{\partial}{\partial h}(\sigma^2 w(h))\right]\frac{1}{\sigma^4 w^2(h)} \\ &= \left[\sigma^2 w(h)\left[2bw'(h) + 2bhw''(h)\right] - 2bh\sigma^2 (w'(h))^2\right]\frac{1}{\sigma^4 w^2(h)}. \end{aligned} \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we have

$$hw''(h) - \left[(\frac{\kappa}{b} - 1) - h(\frac{\rho\sigma x i}{b})\right]w'(h) - \left[\frac{x^2\sigma^2 h}{4b^2} + \frac{ix\sigma^2}{4b^2}\right]w(h) = 0. \tag{3.16}$$

The ODE (3.16) has a general solution of the form [10],

$$w(h) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}h}\left[C_1\Phi(a^*, b^*, \frac{h}{\zeta}) + C_2 h^{1-b^*}\Phi(a^* - b^* + 1, 2 - b^*, \frac{l}{\zeta})\right], \tag{3.17}$$

where

$$a^* = \frac{(\sqrt{\rho^2 - 1} + \rho)\frac{b^*}{2} + \frac{\sigma}{4b}}{\sqrt{\rho^2 - 1}}$$

$$b^* = 1 - \frac{\kappa}{b},$$

and

$$\zeta = \frac{-b}{\sigma x\sqrt{1-\rho^2}}.$$

Here C_1 and C_2 are constants to be determined from the boundary conditions. $\Phi(a, b, z)$ is the degenerated hypergeometric function which has the following Kummer's series expansion

$$\Phi(a, b, z) = 1 + \sum_{k=1}^{\infty}\frac{(a)_k z^k}{(b)_k k!},$$

where

$$(a)_k = a(a+1)\cdots(a+k-1).$$

If we let $C_1 = 1$ and $C_2 = 0$ in (3.17) then a particular solution for (3.16) is

$$w(h) = e^{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}h}\left[\Phi(a^*, b^*, \frac{h}{\zeta})\right].$$

Using the transformation (3.14), Wong and Lo [11] show that a particular solution for (3.13) is

$$U(h) = \frac{2bh}{\sigma^2}\frac{(\sqrt{1-\rho^2}-\rho i)\frac{\sigma x}{2b}\Phi(a^*, b^*, \frac{h}{\zeta}) + \frac{a^*}{b^*\zeta}\Phi(a^*+1, b^*+1, \frac{h}{\zeta})}{\Phi(a^*, b^*, \frac{h}{\zeta})},$$

which can be used to obtain the general solution for (3.13) as follows

$$\hat{D}(h) = U(h) + \frac{\frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{h}{\zeta})}h^{\frac{\kappa}{b}}e^{-2(\sqrt{1-\rho^2})\frac{\sigma x}{2b}(h-1)}}{-\frac{1}{U(1)} + \frac{\sigma^2}{2b}\int_1^h \frac{\Phi^2(a^*,b^*,\frac{1}{\zeta})}{\Phi^2(a^*,b^*,\frac{h}{\zeta})}\eta^{\frac{\kappa}{b}-1}e^{-2(\sqrt{1-\rho^2})\frac{\sigma x}{2b}(\eta-1)}d\eta}. \quad (3.18)$$

We now consider the final ordinary differential equation (3.7). Substituting (3.18) and (3.10) in (3.7), we have

$$\begin{aligned} B_t(t,T) = {} & (\lambda^S m - ba)ixe^{-b(T-t)} - \kappa\theta D(t,T) \\ & - \lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy - \lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz. \end{aligned}$$

Integrating both sides of the above equation and invoking the condition $B(T,T) = 0$, we obtain

$$\begin{aligned} B(t,T) = {} & \left(\frac{\lambda^S m}{b} - a\right) ix(e^{-b(T-t)} - 1) - \kappa\theta \int_t^T D(s,T)ds \\ & + (T-t)\lambda^S \int_{\Re} [e^{ixy} - 1]\phi_Y(y)dy \\ & + (T-t)\lambda^v \int_{\Re} [e^{zD} - 1]\phi_Z(z)dz. \end{aligned} \quad (3.19)$$

□

We can conclude that the characteristic function of the mean reverting process (2.3) with stochastic volatility (2.2) is

$$f(x : t, l, v) = e^{B(t,T)+C(t,T)x+D(t,T)v+ixl},$$

where $B(t,T), C(t,T)$ and $D(t,T)$ are as given in the Lemma.

4 A formula for European Option Pricing

Let C denote the price at time t of a European style call option on the current price of the underlying asset S_t with strike price K and expiration time T.

The terminal payoff of a European call option on the underlying stock price S_t with strike price K is

$$\max(S_T - K, 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \leq K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate r is constant over the lifetime of the option, the price of the European call at time t is equal to the discounted conditional expected payoff

$$C(t, S_T) = e^{-r(T-t)} E_{\mathcal{M}}[\max(S_T - K, 0)|\mathcal{F}_t].$$

Assume that $t = 0$ and we define $L_T = \ln S_T$ and $k = \ln K$. Moreover, we express the call price option $C(0, S_T)$ as a function of the log of the strike price K rather than the terminal log asset price S_T. The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(l)$ by

$$C_T(k) = e^{-rT} \int_k^{\infty} (e^l - e^k) q_T(l) dl, \tag{4.1}$$

where $q_T(l)$ is the density function of the random variable L_T. It was mentioned by Carr and Madan [12] that $C_T(k)$ is not square integrable. To obtain a square integrable function, they introduced the modified call price function $c_T(k)$ defined by

$$c_T(k) = e^{\alpha k} C_T(k) \tag{4.2}$$

for some constant $\alpha > 0$ that makes $c_T(k)$ is square integrable in k over the entire real line and a good choice of α is that one fourth of the upper bound $E[S_T^{\alpha+1}] < \infty$. Consider the Fourier transform of $c_T(k)$

$$\begin{aligned}
\psi_T(u) &= \int_{-\infty}^{\infty} e^{iuk} c_T(k) dk \\
&= \int_{-\infty}^{\infty} e^{iuk} \int_k^{\infty} e^{\alpha k} e^{-rT} (e^l - e^k) q_T(l) dl dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \int_{-\infty}^{l} (e^{l+\alpha k} - e^{(1+\alpha)k}) e^{iuk} dk dl \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(l) \left[\frac{e^{(\alpha+1+iu)l}}{\alpha + iu} - \frac{e^{(\alpha+1+iu)l}}{\alpha + iu + 1} \right] dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{(\alpha + iu) e^{(\alpha+1+iu)l} + e^{(\alpha+1+iu)l} - (\alpha + iu) e^{(\alpha+1+iu)l}}{(\alpha + iu)(\alpha + iu + 1)} \right] q_T(l) dl \\
&= e^{-rT} \int_{-\infty}^{\infty} \left[\frac{e^{(\alpha+1+iu)l}}{\alpha^2 + 2\alpha iu - u^2 + \alpha + iu} \right] q_T(l) dl
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-rT}}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\int_{-\infty}^{\infty} e^{(\alpha+1+iu)l} q_T(l)dl \\
&= \frac{e^{-rT}}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}\int_{-\infty}^{\infty} e^{i(u-(\alpha+1)i)l} q_T(l)dl \\
&= \frac{e^{-rT} f(x=u-(\alpha+1)i : t,l,v)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}
\end{aligned}
$$

where f is the characteristic function defined in Lemma 3.1.
Hence, the European call prices at time $t=0$ with strike price $k=\ln K$ can then be numerically obtained by using the inverse transform:

$$
\begin{aligned}
C_T(k) &= \frac{e^{-\alpha k}}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\psi_T(u)du \\
&= \frac{e^{-\alpha k}}{\pi}\int_{0}^{\infty} e^{-iuk}\frac{e^{-rT} f(x=u-(\alpha+1)i : t,l,v)}{\alpha^2+\alpha-u^2+i(2\alpha+1)u}du. \qquad (4.3)
\end{aligned}
$$

Integration (4.3) is a direct Fourier transform and lends itself to an application of the Fast Fourier Transform (FFT).

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