Option Pricing under a Mean Reverting Process with Jump-Diffusion and Jump Stochastic Volatility

Nonthiya Makate and Pairote Sattayatham\textsuperscript{1}

School of Mathematics, Institute of Science
Suranaree University of Technology, Thailand
e-mail: nonthiyan@hotmail.com (N. Makate)
    pairote@sut.ac.th (P. Sattayatham)

Abstract : An alternative option pricing model is proposed, in which the asset prices follow the jump-diffusion and exhibits mean reversion. The stochastic volatility follows the jump-diffusion with mean reversion. We find a formulation for the European-style option in terms of characteristic functions.

Keywords : jump-diffusion model; stochastic volatility; characteristic function; option pricing; mean reverting.

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1 Introduction

Let $\Omega, \mathcal{F}, \mathbb{P}$ be a probability space with filtration $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. All processes that we shall consider in this section will be defined in this space. An asset price model with stochastic volatility has been defined by Heston \cite{1} which has the following dynamics:

\begin{align*}
    dS_t &= S_t(\mu dt + \sqrt{v_t}dW^S_t), \\
    dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW^v_t,
\end{align*}

where $S_t$ is the asset price, $\mu \in \mathbb{R}$ is the rate of return of the asset price, $v_t$ is the volatility of asset returns, $\kappa > 0$ is the rate at which the volatility reverts

\textsuperscript{1}Corresponding author.

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toward its long-term mean, $\theta \in \mathbb{R}$ is the mean long-term volatility, $\sigma > 0$ is the volatility of the volatility process, $W^S_t$ and $W^v_t$ are standard Brownian motions corresponding to the processes $S_t$ and $v_t$, respectively, with constant correlation $\rho$. Bate [2] introduced the jump-diffusion stochastic volatility model by adding log normal jump $Y_t$ to the Heston stochastic volatility model. In the original formulation of Bate, the model has the following form:

$$dS_t = S_t(\mu dt + \sqrt{v_t}dW^S_t) + S_t - Y_t dN^S_t,$$

where $N^S_t$ is the Poisson process which corresponds to the underlying asset $S_t$, $Y_t$ is a proportion of jump size of the asset price (1.1) with log normal distribution and $S_t -$ means that there is a jump in the value of the process before the jump is used on the left-hand side of the formula. Eraker et al. [3] extended Bate’s work by incorporating jumps into the volatility model, i.e.

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW^v_t + Z_t dN^v_t$$

Eraker et al. [3] developed a likelihood-based estimation strategy and provided estimates of parameters, spot volatility, jump times, and jump sizes using S&P 500 and Nasdaq 100 index returns. Moreover, they examined the volatility structure of the S&P and Nasdaq indices and indicated that models with jumps in volatility are preferred over those without jumps in volatility. But they did not provide a closed-form formula for the price of a European call option.

Empirical evidence on mean reversion in financial assets has been produced by Cecchetti et al. [4] and Bessembinder et al. [5], respectively. It has been documented that currency exchange rates also exhibit mean reversion. Jorion and Sweeney [6] show how the real exchange rates revert to their mean levels and Sweeney [7] provides empirical evidence of mean reversion in G-10 nominal exchange rates. Mean reversion also appears in some stock prices as evidenced by Poterba and Summers [8].

In this paper, we consider the problem of finding a closed-form formula for a European call option where the asset price follows mean reverting jump-diffusion and the stochastic volatility with jump.

The rest of this paper is organized as follows. In Section 2, we briefly discuss model descriptions for option pricing. Deriving a formula for a characteristic function is presented in Section 3. Finally, a closed-form formula for a European call option in terms of characteristic functions is presented.

## 2 Model Descriptions

It is assumed that a risk-neutral probability measure $\mathcal{M}$ exists. The asset price $S_t$ under this measure follows a mean reverting jump-diffusion process, and the volatility $v_t$ follows mean reverting with jump, i.e. our models are governed
by the following dynamics:

\[
\begin{align*}
  dS_t &= b \left( a - \ln S_t - \frac{\lambda S m}{b} \right) dt + \sqrt{v_t} S_t dW^S_t + S_t \left( Y_t - Y_t dN^S_t \right) \\
  dv_t &= \kappa (\theta - v_t) dt + \sigma \sqrt{v_t} dW^v_t + Z_t dN^v_t
\end{align*}
\]  

(2.1)  

(2.2)

where \( S_t, v_t, \kappa, \theta, \sigma, W^S_t \) and \( W^v_t \) are defined as above, \( a \in \mathbb{R} \) is the mean of long-term asset price return, \( b > 0 \) is the rate at which the asset price return reverts toward its long-term mean, \( N^S_t \) and \( N^v_t \) are independent Poisson processes with constant intensities \( \lambda^S \) and \( \lambda^v \) respectively. \( Y_t \) and \( Z_t \) are proportional jump sizes of the asset price (2.1) and the jump size of the volatility process (2.2) respectively. Suppose that \( Y_t \) and \( Z_t \) are independent and identically distributed sequences with densities \( \phi_Y(y) := \phi_Y(y) \), \( \phi_Z(z) := \phi_Z(z) \) and \( EY_t := m \). Moreover, we assume that the jump processes \( N^S_t \) and \( N^v_t \) are independent of standard Brownian motions \( W^S_t \) and \( W^v_t \).

Assume that the asset price \( S_t \) and the volatility \( v_t \) satisfy equations (2.1) and (2.2) respectively. Let \( L_t = \ln S_t \), by the jump-diffusion chain rule, \( \ln S_t \) satisfies the SDE

\[
  dL_t = \left( a - L_t - \frac{\lambda S m}{b} - \frac{v_t}{2b} \right) dt + \sqrt{v_t} dW^S_t + \ln(1 + Y_t) dN^S_t.
\]  

(2.3)

### 3 Characteristic Functions

We denote the characteristic function for \( L_T = \ln S_T \) as

\[
  f(x : t, l, v) = E_M[e^{ixL_T} | L_t = l, v_t = v]
\]  

(3.1)

where \( 0 \leq t \leq T \) and \( i = \sqrt{-1} \). Here \( L_t \) is the mean reverting asset price process with jumps specified by (2.3) and \( v_t \) is the volatility process specified by (2.2). The generalized Feynman-Kac theorem [9] implies that \( f(x : t, l, v) \) solves the following partial integro-differential equation (PIDE):

\[
  0 = \frac{\partial f}{\partial t} + b \left( a - l - \frac{\lambda S m}{b} - \frac{v}{2b} \right) \frac{\partial f}{\partial l} + \kappa (\theta - v) \frac{\partial f}{\partial v} + \frac{1}{2} \sigma^2 v \frac{\partial^2 f}{\partial v^2} + \rho \sigma v \frac{\partial^2 f}{\partial l \partial v} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial v^2}
\]

\[
  + \lambda^S \int_{\mathbb{R}} [f(x : t, l + y, v) - f(x : t, l, v)] \phi_Y(y) dy
\]

\[
  + \lambda^v \int_{\mathbb{R}} [f(x : t, l, v + z) - f(x : t, l, v)] \phi_Z(z) dz.
\]  

(3.2)
Lemma 3.1. Suppose that $L_t$ follows the dynamics in (2.3). Then the characteristic function for $L_T$ can be written in the form
\[
f(x : t, l, v) = \exp[B(t, T) + C(t, T)l + D(t, T)v + ixl],
\]
(3.3)
where
\[
B(t, T) = (\frac{\lambda S_m}{b} - a)ix(e^{-b(T-t)} - 1) - \theta \kappa \int_t^T D(s, T)ds
\]
\[
+ (T - t)\lambda^S \int_0^\infty [e^{ixy} - 1] \phi_Y(y)dy
\]
\[
+ (T - t)\lambda^C \int_0^\infty [e^{-D(t,T)} - 1] \phi_Z(z)dz,
\]
\[
C(t, T) = ix(e^{-b(T-t)} - 1),
\]
\[
D(t, T) = U(e^{-b(T-t)}) + \frac{e^{-\kappa(T-t)}V(e^{-b(T-t)})}{-\frac{1}{U(t)} + \frac{\sigma^2}{2b} \int_1^{e^{-b(T-t)}} h^{-1}V(h)dh},
\]
\[
U(h) = \frac{2bh(\sqrt{1 - \rho^2} - \rho i)\Phi(a^*, b^*, \frac{h}{\zeta})}{\Phi^2(a^*, b^*, \frac{1}{\zeta})},
\]
\[
V(h) = \frac{\Phi^2(a^*, b^*, \frac{1}{\zeta})e^{(\sqrt{1 - \rho^2})\frac{\pi}{2}(1 - h)}}{\Phi^2(a^*, b^*, \frac{1}{\zeta})},
\]
\[
h = e^{-b(T-t)},
\]
\[
a^* = \frac{b^*}{2}(\sqrt{\rho^2 - 1} + \rho) + \frac{\sigma^2}{4\rho^2 - 1},
\]
\[
b^* = 1 - \frac{\kappa}{\gamma},
\]
\[
\zeta = -\frac{b}{\sigma x \sqrt{1 - \rho^2}}
\]
and $\Phi(\cdot,\cdot,\cdot)$ is the degenerated hypergeometric function.

Proof. From (3.1), it is clear that
\[
f(x : t, l, v) = e^{ixl}
\]
(3.4)
which is the boundary condition of PIDE (3.2). This implies that
\[
B(T, T) = C(T, T) = D(T, T) = 0.
\]
(3.5)
Substituting (3.3) in (3.2) and using the fact that the function $f$ is never zero, we obtain
\begin{align*}
0 &= [B_t + (ba - \lambda^S m)(C + ix) + \kappa\theta D \\
&\quad + \lambda^S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y) dy + \lambda^v \int_\mathbb{R} [e^{zD} - 1] \phi_Z(z) dz \\
&\quad + [C_t - b(C + ix)] l \\
&\quad + [D_t + \frac12 (C + ix) + \frac12 (C + ix)^2 - \kappa D + \frac12 \sigma^2 D^2 + \rho \sigma (C + ix) D] v \\
&\quad + \lambda S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y) dy + \lambda^v \int_\mathbb{R} [e^{zD} - 1] \phi_Z(z) dz ] + [C_t - b(C + ix)] l \\
&\quad + [D_t + 1 \frac12 (C + ix) + \frac12 (C + ix)^2 - \kappa D + \frac12 \sigma^2 D^2 + \rho \sigma (C + ix) D] v \\
\end{align*}

where \(B_t, C_t\) and \(D_t\) are the partial derivatives with respect to \(t\) of functions \(B, C\) and \(D\) respectively.

This reduces the problem to one of solving three, much simpler, ordinary differential equations:

\begin{align*}
B_t + (ba - \lambda^S m)(C + ix) + \kappa\theta D + \lambda^S \int_\mathbb{R} [e^{ixy} - 1] \phi_Y(y) dy \\
&\quad + \lambda^v \int_\mathbb{R} [e^{zD} - 1] \phi_Z(z) dz = 0 \\
C_t - b(C + ix) &= 0 \\
D_t + \frac12 (C + ix)(C + ix - 1) - \kappa D + \frac12 \sigma^2 D^2 + \rho \sigma (C + ix) D &= 0
\end{align*}

subject to boundary conditions (3.5).

The solution to equation (3.8) with the boundary condition \(C(T, T) = 0\) is given by

\(C(t, T) = ix(e^{-b(T-t)} - 1).\) (3.10)

We now consider equation (3.9). Substituting (3.10) in (3.9), one gets

\(D_t + \frac12 [ixe^{-b(T-t)}] [ixe^{-b(T-t)} - 1] - \kappa D + \frac12 \sigma^2 D^2 + \rho \sigma ix De^{-b(T-t)} = 0.\)

Hence,

\(D_t = -\frac12 \sigma^2 D^2 + \left[\kappa - \rho \sigma ix e^{-b(T-t)}\right] D + \frac12 \left[\sigma^2 e^{-2b(T-t)} + i x e^{-b(T-t)}\right].\) (3.11)

Let \(h = e^{-b(T-t)}\) and we define a new function \(\hat{D}(h(t), T) := D(t, T).\) Then

\[\frac{\partial D(t, T)}{\partial t} = \frac{\partial \hat{D}(h, T)}{\partial h} \frac{\partial h}{\partial t}
\]

\[= be^{-b(T-t)} \frac{\partial \hat{D}(h, T)}{\partial h}.\] (3.12)
Substituting (3.12) into (3.11), we obtain the following Riccati equation

\[
\frac{\partial \hat{D}}{\partial h} = -\frac{1}{2bh}\sigma^2 \hat{D}^2 + \left(\frac{\kappa}{bh} - \frac{\rho \sigma ix}{b}\right) \hat{D} + \frac{1}{2b}(x^2 h + ix) .
\] (3.13)

We shall solve the second order ODE (3.13) together with the initial condition \( \hat{D}(1, T) = 0 \). Let

\[
\hat{D}(h, T) = \frac{2bhw'(h)}{\sigma^2 w(h)}
\] (3.14)

and taking the derivative of (3.14) with respect to \( h \), one gets

\[
\frac{\partial \hat{D}}{\partial h} = \left[\sigma^2 w(h) \frac{\partial}{\partial h}(2bh w'(h)) - 2bh w'(h) \frac{\partial}{\partial h}(\sigma^2 w(h))\right] \frac{1}{\sigma^4 w^2(h)}
\] (3.15)

Substituting (3.14) and (3.15) into (3.13), we have

\[
h w''(h) - \left(\frac{\kappa}{b} - 1\right) - h(\frac{\rho \sigma ix}{b}) w'(h) - \left[\frac{x^2 \sigma^2 h}{4b^2} + \frac{ix \sigma^2}{4b^2}\right] w(h) = 0
\] (3.16)

The ODE (3.16) has a general solution of the form [10],

\[
w(h) = e^{(\sqrt{1-\rho^2} - \rho)i h} \left[C_1 \Phi(a^*, b^*, \frac{h}{\zeta}) + C_2 h^{1-b^*} \Phi(a^* - b^* + 1, 2 - b^*, \frac{h}{\zeta})\right],
\] (3.17)

where

\[
a^* = \frac{\sqrt{\rho^2 - 1} + \rho}{\sqrt{\rho^2 - 1}} + \frac{\sigma i}{\sigma x \sqrt{1 - \rho^2}}
\]

\[
b^* = 1 - \frac{\kappa}{b}
\]

and

\[
\zeta = \frac{-b}{\sigma x \sqrt{1 - \rho^2}}.
\]

Here \( C_1 \) and \( C_2 \) are constants to be determined from the boundary conditions. \( \Phi(a, b, z) \) is the degenerated hypergeometric function which has the following Kummer's series expansion

\[
\Phi(a, b, z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k z^k}{(b)_k k!},
\]

where

\[(a)_k = a(a + 1) \cdots (a + k - 1).\]
If we let $C_1 = 1$ and $C_2 = 0$ in (3.17) then a particular solution for (3.16) is

$$w(h) = e^{(\sqrt{1 - \rho^2} - \rho i) \frac{S}{T} h} \Phi(a^*, b^*, \frac{h}{\zeta}).$$

Using the transformation (3.14), Wong and Lo [11] show that a particular solution for (3.13) is

$$U(h) = \frac{2bh}{\sigma^2} \sqrt{1 - \rho^2} \Phi(a^*, b^*, \frac{h}{\zeta}) + \Phi(a^* + 1, b^* + 1, \frac{h}{\zeta}),$$

which can be used to obtain the general solution for (3.13) as follows

$$\hat{D}(h) = U(h) + \frac{\Phi_0^2(a^*, b^*, \frac{1}{\zeta})}{\Phi^2_0(a^*, b^*, \frac{1}{\zeta})} \Phi_0 e^{-2(\sqrt{1 - \rho^2} \frac{S}{T} h)(h - 1)}.$$

(3.18)

We now consider the final ordinary differential equation (3.7). Substituting (3.18) and (3.10) in (3.7), we have

$$B_t(t, T) = (\lambda S m - ba)ixe^{-b(T-t)} - \kappa \theta D(t, T)$$

$$- \lambda \int_{\mathbb{R}} [e^{ixy} - 1] \phi_Y(y) dy - \lambda \int_{\mathbb{R}} [e^{izD} - 1] \phi_Z(z) dz.$$

Integrating both sides of the above equation and invoking the condition $B(T, T) = 0$, we obtain

$$B(t, T) = \left( \frac{\lambda S m}{b} - a \right) ix(e^{-b(T-t)} - 1) - \kappa \theta \int_t^T D(s, T) ds$$

$$+ (T - t) \lambda S \int_{\mathbb{R}} [e^{ixy} - 1] \phi_Y(y) dy$$

$$+ (T - t) \lambda \int_{\mathbb{R}} [e^{izD} - 1] \phi_Z(z) dz. \quad (3.19)$$

We can conclude that the characteristic function of the mean reverting process (2.3) with stochastic volatility (2.2) is

$$f(x : t, l, v) = e^{B(t, T) + C(t, T)x + D(t, T)v + ixl},$$

where $B(t, T), C(t, T)$ and $D(t, T)$ are as given in the Lemma.
4 A formula for European Option Pricing

Let $C$ denote the price at time $t$ of a European style call option on the current price of the underlying asset $S_t$ with strike price $K$ and expiration time $T$.

The terminal payoff of a European call option on the underlying stock price $S_t$ with strike price $K$ is

$$\max(S_T - K, 0).$$

This means that the holder will exercise his right only if $S_T > K$ and then his gain is $S_T - K$. Otherwise, if $S_T \leq K$, then the holder will buy the underlying asset from the market and the value of the option is zero.

Assuming the risk-free interest rate $r$ is constant over the lifetime of the option, the price of the European call at time $t$ is equal to the discounted conditional expected payoff

$$C(t, S_T) = e^{-r(T-t)}E_M[\max(S_T - K, 0)|\mathcal{F}_t].$$

Assume that $t = 0$ and we define $L_T = \ln S_T$ and $k = \ln K$. Moreover, we express the call price option $C(0, S_T)$ as a function of the log of the strike price $K$ rather than the terminal log asset price $S_T$. The initial call value $C_T(k)$ is related to the risk-neutral density $q_T(l)$ by

$$C_T(k) = e^{-rT} \int_{k}^{\infty} (e^l - e^k)q_T(l)dl, \quad (4.1)$$

where $q_T(l)$ is the density function of the random variable $L_T$. It was mentioned by Carr and Madan [12] that $C_T(k)$ is not square integrable. To obtain a square integrable function, they introduced the modified call price function $c_T(k)$ defined by

$$c_T(k) = e^{\alpha k}C_T(k) \quad (4.2)$$

for some constant $\alpha > 0$ that makes $c_T(k)$ is square integrable in $k$ over the entire real line and a good choice of $\alpha$ is that one fourth of the upper bound $E[S_T^{\alpha+1}] < \infty$. Consider the Fourier transform of $c_T(k)$

$$\psi_T(u) = \int_{-\infty}^{\infty} e^{iku}c_T(k)dk$$

$$= \int_{-\infty}^{\infty} e^{iku} \int_{k}^{\infty} e^{\alpha k} e^{-rT}(e^l - e^k)q_T(l)dl dk$$

$$= \int_{-\infty}^{\infty} e^{-rT}q_T(l) \int_{-\infty}^{l} (e^{l+\alpha k} - e^{(1+\alpha)k})e^{iku}dl$$

$$= \int_{-\infty}^{\infty} e^{-rT}q_T(l) \left[ \frac{e^{(\alpha+1+iu)l} - e^{(\alpha+1+iu)\alpha}}{\alpha + iu - (\alpha + iu)} \right] dl$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left[ \frac{(\alpha + iu)e^{(\alpha+1+iu)\alpha} + e^{(\alpha+1+iu)l} - (\alpha + iu)e^{(\alpha+1+iu)\alpha}}{(\alpha + iu)(\alpha + iu + 1)} \right] q_T(l)dl$$

$$= e^{-rT} \int_{-\infty}^{\infty} \left[ \frac{e^{(\alpha+1+iu)l}}{\alpha^2 + 2\alpha iu - u^2 + \alpha + iu} \right] q_T(l)dl$$
where $f$ is the characteristic function defined in Lemma 3.1. Hence, the European call prices at time $t = 0$ with strike price $k = \ln K$ can then be numerically obtained by using the inverse transform:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-iku} \psi_T(u) du$$

$$= \frac{e^{-\alpha k}}{\pi} \int_{0}^{\infty} e^{-iku} \frac{e^{-rT} f(x = u - (\alpha + 1)i : t, l, v)}{\alpha^2 + \alpha - u^2 + i(2\alpha + 1)u} du. \tag{4.3}$$

Integration (4.3) is a direct Fourier transform and lends itself to an application of the Fast Fourier Transform (FFT).

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