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CONTROLLING FOR A DISCRETE-TIME SURPLUS PROCESS IN INSURANCE TO REACH A FIRM'S DESIRED TARGET

K. CHUARKHAM, P. SATTAYATHAM* and W. KLONGDEE†

School of Mathematics
Suranaree University of Technology
Nakhon Ratchasima 30000, Thailand
e-mail: pairote@sut.ac.th

†Applied Mathematics Research Group
Department of Mathematics
Faculty of Science
Khonkaen University
40002, Thailand

Abstract

In this paper, we consider a discrete-time surplus process of the form

$$X_0 = x, \quad X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1}, \quad n = 0, 1, 2, \ldots$$

Reinsurance and shareholder input may be considered as control parameters which allow a firm to reach a desired target. We prove the existence of an optimal plan and we obtain a formula for the valued function which gives an optimal control policy. An example shows some numerical calculations for getting an optimal plan.

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*Corresponding author

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1. Introduction

In recent years, risk models have attracted much attention in the insurance business, in connection with any possible insolvency and the capital reserves of the insurance company. The main interest from the point of view of an insurance company is claim arrival and claim size, which affect the capital of the company.

In this paper, we assume that all processes are defined in a probability space $(\Omega, \Im, P)$. Claims happen at the times T_i, satisfying $0 = T_0 \le T_1 \le T_2 \le \cdots$. We call them *claim arrivals* or, simply, *arrivals*. The nth claim arriving at time T_n causes the *claim size* Y_n. The *inter-arrival times* or, simply, *inter-arrival*, $Z_n := T_n - T_{n-1}$ is the length of time between the $(n-1)$th claim and the nth claim. By *period* n, we shall mean the random interval $[T_{n-1}, T_n)$, $n \ge 1$.

Now let the constant c_0 represent the premium rate for one unit time; the random variable $c_0 \sum_{i=1}^{n+1} Z_i = c_0 T_{n+1}$ describes the inflow of capital into the business in $[0, T_{n+1}]$, and $\sum_{i=1}^{n+1} Y_i$ describes the outflow of capitals due to payments for claims occurring in $[0, T_{n+1}]$. Therefore, the quantity

$$X_{n+1} = x + c_0 T_{n+1} - \sum_{i=1}^{n+1} Y_i, \; n = 0, 1, 2, \ldots \tag{1}$$

is the insurer's balance (or surplus) at time T_{n+1} with the constant $x \ge 0$ as initial capital.

In summary, the discrete-time surplus process (1) can be written in the form:

$$X_0 = x, \quad X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1}, \quad n = 0, 1, 2, \ldots. \tag{2}$$

Usually, this model was considered under the assumption that $\{Y_n, n \ge 1\}$ and $\{Z_n, n \ge 1\}$ are independent. In 2004, Schäl [10] applied this model with reinsurance and investment are control parameters and proved the existence of an optimal plan for the exponential utility function under the assumption of independent as mentioned above. Recently, Klongdee et al. [6] applied this model with reinsurance and investment as control parameters and proved the existence of the optimal plan for the exponential utility function under the additional assumption that a reinsurer has the opportunity to default.

In this paper, we study this model together with two controllers, i.e., reinsurance and shareholder input (i.e., the amount of money that the shareholders put into the firm) allowing the firm to reach a desired target. Moreover, we find an optimal control policy which minimizes a reasonable objective function.

2. Model Description

Let $\{X_n, n \geq 0\}$ be the surplus process which can be controlled by choosing a retention level $b \in [\underline{b}, \overline{b}]$, $0 \leq \underline{b} \leq b \leq \overline{b} \leq \infty$, of a reinsurance for one period. Next, for each level b, an insurer has to pay a premium rate to a reinsurer which is deducted from c_0. As a result, the insurer's income rate will be represented by a function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level, which can be chosen. Of course, we obtain the *net income rate* $c(b)$, where $0 \leq c(b) \leq c_0$ for all $b \in [\underline{b}, \overline{b}]$ and $c(b)$ is non-decreasing. By the *expected value principle*, c_0 and $c(b)$ can be calculated as follows:

$$c_0 = (1 + \theta_0)\frac{E[Y]}{E[Z]} \quad \text{and} \quad c(b) = c_0 - (1 + \theta_1)\frac{E[Y - h(b, Y)]}{E(Z)}, \tag{3}$$

where Y is a claim size, Z is an inter-arrival time, and $0 < \theta_0 < 1$, $0 < \theta_1 < 1$ are the *safety loading* of the insurer and reinsurer, respectively. The measurable function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which is called *reinsurance recovery* paid by the reinsurer. In the case of an *excess of loss reinsurance*, we have

$$h(b, y) = \min\{b, y\} \text{ with retention level } 0 \leq \underline{b} \leq b \leq \overline{b} = \infty.$$

In the case of a *proportional reinsurance*, we have

$$h(b, y) = by \text{ with retention level } 0 \leq \underline{b} \leq b \leq \overline{b} = 1.$$

Furthermore, the surplus process can also be controlled by shareholder input, i.e., the insurance company can ask its shareholders to input their money $\delta \in [0, \infty)$, so that the firm can reach a desired target A. Hence the two controllers for the surplus process, b and δ, will stand for reinsurance and shareholder input, respectively. Note that, we can interpret the target A as an initial capital for supporting the growth of various policies in the future.

Let b_n and δ_n be the two control actions at the time T_n. Therefore, the surplus process (2) can be modified to be the following:

$$X_{n+1} = X_n + \delta_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1}),\ n = 0, 1, 2, \ldots, \tag{4}$$

where $X_0 = x$. It is convenient to rewrite (4) into an equivalent form

$$X_{n+1} = X_n + L(b_n, \delta_n, Y_{n+1}, Z_{n+1}),\ n = 0, 1, 2, \ldots, \tag{5}$$

where $L(b, \delta, y, z) = \delta + c(b)z - h(b, y)$. We see that the process $\{X_n, n \geq 0\}$ is driven by the control actions $\{(b_n, \delta_n), n \geq 0\}$, the sequence of inter-arrival times $\{Z_n, n \geq 1\}$, and the sequence of claims $\{Y_n, n \geq 1\}$. Let us assume that $\{Z_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent and identically distributed (iid) sequences of random variables with finite variance, i.e., we make the following assumption:

Assumption 1. Independence Assumption (IA)

The sequence of inter-arrival times $\{Z_n, n \geq 1\}$ and the sequence of claims $\{Y_n, n \geq 1\}$ are iid sequences with finite variances. Moreover, for each $n \in \{1, 2, 3, \ldots\}$, Z_n and Y_n are independent.

We immediately get from Assumption 1 that $\{h(b_n, Y_{n+1}), n \geq 0\}$ is an independent sequence.

Remark 1. *Let* $n \in \{0, 1, 2, \ldots\}$ *and f be the density of* Y_{n+1}. *Then the variance of* $h(b_n, Y_{n+1})$ *is finite.*

Proof. We consider

$$Var[h(b_n, Y_{n+1})]$$

$$= E[h^2(b_n, Y_{n+1})] - (E[h(b_n, Y_{n+1})])^2$$

$$= E[b_n^2 1_{Y_{n+1} > b_n} + Y_{n+1}^2 1_{Y_{n+1} \le b_n}] - (E[b_n 1_{Y_{n+1} > b_n} + Y_{n+1} 1_{Y_{n+1} \le b_n}])^2$$

$$= b_n^2 P[Y_{n+1} > b_n] + \int_{y \le b_n} y^2 f(y) dy - \left(b_n P[Y_{n+1} > b_n] + \int_{y \le b_n} y f(y) dy \right)^2.$$

Since Y_{n+1} has finite variance, $\int_{y \le b_n} y^2 f(y) dy < \infty$ and $\int_{y \le b_n} y f(y) dy < \infty$. Thus $Var[h(b_n, Y_{n+1})] < \infty$ and this proves Remark 1.

3. A Value Function with Finite Horizon

Let $\{X_n, n \ge 0\}$ be a surplus process with value in a state space (S, Ξ) which is a measurable space. The surplus process can be controlled at the beginning of every period $[T_n, T_{n+1})$, $n = 0, 1, 2, \ldots$ on a measurable space $(U, \mho)$, which is called a *control space*. In addition, the model is further specified by the following quantities:

- $N \in \{2, 3, 4, \ldots\}$ is a *time horizon* (number of periods);
- T_N is a *time at the time horizon N*;
- $\alpha_N \in (0, 1]$ is a *positive real constant*;
- $g : S \times U \to (-\infty, \infty)$ is a *one-period cost function*, which is measurable and bounded from below;
- $\hat{V} : S \to (-\infty, \infty]$ is a *cost function* for time horizon N, which is measurable and bounded from below.

Definition 1. A plan for the time horizon N over a control space U is a (finite) sequence $\pi = \{u_n\}_{n=0}^{N-1}$ of $u_0 = (b_0, \delta_0) = (b_0, 0)$ and $u_n = (b_n, \delta_n) \in U$ for $n = 1, 2, 3, \ldots, N-1$. A set of all plans for the time horizon N over the space U is denoted by $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be *stationary*, if $b_0 = b_1$ and $(b_n, \delta_n) = (b_1, \delta_1)$ for $n = 1, 2, 3, \ldots, N-1$.

For each initial state $x \in S$ and plan $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$, the surplus process (5) can be written in the form

$$X_{n+1} = X_n + L(b_n, \delta_n, Y_{n+1}, Z_{n+1})$$

$$= x + \sum_{k=0}^{n} L(b_k, \delta_k, Y_{k+1}, Z_{k+1}), \; n = 0, 1, 2, ..., N-1 \qquad (6)$$

with $X_0 = x$.

Definition 2. Let $x \in S$ be an initial state and $\pi = \{u_n\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$, where N is the time horizon. The total cost function $\Phi_N(x, \pi)$ and the valued function $V_N(x)$ for the time horizon N are defined by

$$\Phi_N(x, \pi) = E\left[\sum_{n=0}^{N-1} g(X_n, u_n) + \alpha_N \hat{V}(X_N) |_{X_0 = x}\right]$$

and

$$V_N(x) = \inf_{\pi \in \mathcal{P}(N, U)} \Phi_N(x, \pi), \text{ respectively,} \qquad (7)$$

when X_n's are random variables which satisfy equation (6). A plan $\tilde{\pi} \in \mathcal{P}(N, U)$ is said to be *optimal*, if $\inf_{\pi \in \mathcal{P}(N, U)} \Phi_N(x, \pi) = \Phi_N(x, \tilde{\pi})$.

4. Main Results

First, we note that it is natural to assume that the target A should satisfy the condition

$$A \geq E[X_N |_{X_0 = x}], \qquad (8)$$

where X_N is a random variable satisfying equation (2). The above expectation can be calculated as follows:

$$E[X_N |_{X_0 = x}] = E[X_{N-1} + c_0 Z_N - Y_N |_{X_0 = x}]$$

$$= x + c_0 \sum_{n=1}^{N} E[Z_n] - \sum_{n=1}^{N} E[Y_n].$$

Since the goal of this paper is to find the retention level and the shareholder input that can make the firm reach a desired target A, the case of shareholder input greater than A is uninteresting. So, we shall assume that

$$S = \mathcal{R} \quad \text{and} \quad U = [\underline{b}, \overline{b}] \times [0, A] \tag{9}$$

are the state space and the control space, respectively.

In this section, we study the surplus model (6) when the insurance company is controlled by choosing the retention level b_n and the shareholder input δ_n at the beginning of the period $[T_n, T_{n+1})$ in order to reach the desired target A at the time horizon N.

We will study the cost function under the assumption that the insurance company is solvent (not ruined) and we will look for a control policy that ensures the minimization of the distance from the surplus at the time horizon N to the target A. Therefore, we define *one-period cost function* and *cost function* at the time horizon N, respectively, as follows:

$$g(x, u) = g(x, (b, \delta)) = \delta^2 \quad \text{and} \quad \hat{V}(x) = (x - A)^2,$$

where $u = (b, \delta) \in U$ and $x \in S$. Thus, we obtain the *total cost function* of model (7) as

$$\Phi_N(x, \pi) = \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E[(X_N - A)^2 |_{X_0 = x}], \tag{10}$$

where $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$.

Remark 2. By substituting $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ into equation (10), we get

$$\Phi_N(x, \pi) = \sum_{n=1}^{N-1} \delta_n^2$$
$$+ \alpha_N \left\{ \sum_{n=0}^{N-1} \{c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\} + G_N^2(x, \pi) \right\}, \tag{11}$$

where $G_N(x, \pi) = x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})]$.

Proof. Let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$. Then

$$\Phi_N(x, \pi) = \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E[(X_N - A)^2 |_{X_0 = x}]$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E\left(x + \sum_{n=0}^{N-1} L(b_n, \delta_n, Y_{n+1}, Z_{n+1}) - A\right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N Var\left(x + \sum_{n=0}^{N-1} L(b_n, \delta_n, Y_{n+1}, Z_{n+1}) - A\right)$$

$$+ \alpha_N\left(x + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})] - A\right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N Var\left(x - A + \sum_{n=0}^{N-1} \{\delta_n + c(b_n) Z_{n+1} - h(b_n, Y_{n+1})\}\right)$$

$$+ \alpha_N\left(x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})]\right)^2$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N \left\{\sum_{n=0}^{N-1} \{c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\} + G_N^2(x, \pi)\right\},$$

where $G_N(x, \pi) = x - A + \sum_{n=0}^{N-1} E[L(b_n, \delta_n, Y_{n+1}, Z_{n+1})]$.

Remark 3. Define a subset $\mathcal{P}^*(N, U)$ of $\mathcal{P}(N, U)$ by

$$\mathcal{P}^*(N, U) = \{\pi \in \mathcal{P}(N, U) \mid G_N(x, \pi) = 0\}.$$

We have

(i) $\mathcal{P}^*(N, U)$ is not empty.

(ii) $\mathcal{P}^*(N, U)$ contains an element of the form $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1}$, where $\delta_1 = \delta_2 = \cdots = \delta_{N-1}$.

Proof of (i). We choose an arbitrary finite sequence $b_n \in [\underline{b}, \overline{b}]$, $n = 0, 1, 2, \ldots, N-1$ and $\delta_0 = 0$, $\delta_n = \frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} - c(b_n)E[Z_{n+1}] + E[h(b_n, Y_{n+1})]$, $n = 1, 2, 3, \ldots, N-1$. From inequality (8), we have

$$A \geq x + \sum_{n=1}^{N} c_0 E[Z_n] - \sum_{n=1}^{N} E[Y_n] = x + \sum_{n=0}^{N} (c_0 E[Z_n] - E[Y_n]).$$

Following from the expected value principle, we have $0 < c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]$ for all n. Moreover, by equation (3), we get

$$\begin{aligned}
0 &< c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})] \\
&= E[Z_{n+1}]\left(c_0 - (1+\theta_1)\frac{E[Y_{n+1} - h(b_n, Y_{n+1})]}{E[Z_{n+1}]}\right) - E[h(b_n, Y_{n+1})] \\
&= c_0 E[Z_{n+1}] - (1+\theta_1)E[Y_{n+1} - h(b_n, Y_{n+1})] - E[h(b_n, Y_{n+1})] \\
&= c_0 E[Z_{n+1}] - (1+\theta_1)E[Y_{n+1}] + \theta_1 E[h(b_n, Y_{n+1})] \\
&= c_0 E[Z_{n+1}] - E[Y_{n+1}] + \theta_1(E[h(b_n, Y_{n+1})] - E[Y_{n+1}]) \\
&\leq c_0 E[Z_{n+1}] - E[Y_{n+1}]
\end{aligned}$$

$$(\text{since } E[h(b_n, Y_{n+1})] \leq E[Y_{n+1}] \text{ and } \theta_1 > 0), \quad n = 0, 1, 2, \ldots, N-1. \tag{12}$$

By taking summation to both sides of inequality (12), we get

$$\begin{aligned}
0 &< x + \sum_{n=0}^{N-1} (c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) \\
&\leq x + \sum_{n=0}^{N-1} (c_0 E[Z_{n+1}] - E[Y_{n+1}]) \leq A.
\end{aligned} \tag{13}$$

Claim that $0 \leq \delta_n \leq A$, $n = 1, 2, \ldots, N-1$.

First, assume that there exists $\delta_m < 0$ for some $m \in \{1, 2, \ldots, N-1\}$. It follows from the definition of δ_m that

$$\frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} < c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})]. \tag{14}$$

Putting $n = 0$ in inequality (12), we get

$$-c_0 E[Z_1] + E[Y_1] \leq -c(b_0)E[Z_1] + E[h(b_0, Y_1)].$$

Hence

$$\frac{A - x - c_0 E[Z_1] + E[Y_1]}{N-1} \leq \frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1}$$

$$< c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})] \text{ (by inequality (14))}$$

$$< c_0 E[Z_{m+1}] - E[Y_{m+1}] \text{ (by inequality (12))}$$

$$= c_0 E[Z_1] - E[Y_1] \text{ (by iid property of } Y_n \text{ and } Z_n).$$

Thus $A < x + N(c_0 E[Z_1] - E[Y_1])$. It follows from inequality (13) that

$$x + N(c_0 E[Z_1] - E[Y_1]) = x + \sum_{n=0}^{N-1}(c_0 E[Z_{n+1}] - E[Y_{n+1}]) \leq A.$$

This is a contradiction and then $\delta_n \geq 0$, $n = 1, 2, \ldots, N-1$.

Next, assume that there exists $\delta_m > A$ for some $m \in \{1, 2, \ldots, N-1\}$. Again, by the definition of δ_m, we have

$$\frac{A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)]}{N-1} > A + c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})]$$

$$> A \text{ (since } c(b_m)E[Z_{m+1}] - E[h(b_m, Y_{m+1})] > 0).$$

Thus

$$A - x - c(b_0)E[Z_1] + E[h(b_0, Y_1)] > (N-1)A.$$

Hence $-(N-2)A > x + c(b_0)E[Z_1] - E[h(b_0, Y_1)] \geq x \geq 0$. This is a contradiction since $-(N-2)A$ is negative and cannot be greater than zero. Therefore, we have the claim and then the plan $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$. Moreover, we can easily see that

$$G_N(x, \pi) = x - A + \sum_{n=1}^{N-1}\delta_n + \sum_{n=0}^{N-1}(c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]) = 0.$$

Then $\mathcal{P}^*(N, U)$ is not empty. This proves (i).

Proof of (ii). By choosing $\delta_0 := 0,$

$$\delta_n := \frac{A - x - \sum_{k=0}^{N-1}(c(b_k)E[Z_{k+1}] - E[h(b_k, Y_{k+1})])}{N-1}, \quad n = 1, 2, \ldots, N-1.$$

Hence $\delta_1 = \delta_2 = \cdots = \delta_{N-1}$. By the same proof as in case (i), we have $0 \le \delta_n \le A$, $n = 0, 1, 2, \ldots, N-1$. Thus the plan $\pi := \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$. Obviously, $G_N(x, \pi) = 0$. Hence the plan $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ and this proves (ii).

Lemma 4. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. Then $\Phi_N(x, \pi) < \Phi_N(x, ((b_0, 0), (b_1, 0), (b_{N-1}, 0)))$.*

Proof. Let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. Hence $G_N(x, \pi) = 0$. It follows from equation (11) and the iid property of $Z_1, Z_2, \ldots, Z_N$ (Assumption 1) that

$$\Phi_N(x, \pi) = \sum_{n=1}^{N-1}\delta_n^2 + \alpha_N\sum_{n=0}^{N-1}\{c^2(b_n)Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\}$$

$$= (N-1)\delta_1^2 + \alpha_N\sum_{n=0}^{N-1}\{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\}.$$

Next, we consider

$$\Phi_N(x, ((b_0, 0), (b_1, 0), \ldots, (b_{N-1}, 0)))$$

$$= \alpha_N E\left(x - A + \sum_{n=0}^{N-1}c(b_n)Z_{n+1} - \sum_{n=0}^{N-1}h(b_n, Y_{n+1})\right)^2$$

$$= \alpha_N Var\left(x - A + \sum_{n=0}^{N-1}\{c(b_n)Z_{n+1} - h(b_n, Y_{n+1})\}\right)$$

$$+ \alpha_N\left(x - A + \sum_{n=0}^{N-1}\{c(b_n)E[Z_{n+1}] - E[h(b_n, Y_{n+1})]\}\right)^2$$

$$= \alpha_N \sum_{n=0}^{N-1} \{c^2(b_n) Var[Z_{n+1}] + Var[h(b_n, Y_{n+1})]\}$$

$$+ \alpha_N \left(x - A + (N-1)\delta_1 - (N-1)\delta_1 + \sum_{n=0}^{N-1} \{c(b_n) E[Z_{n+1}] - E[h(b_n, Y_{n+1})]\} \right)^2$$

$$= \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\} + \alpha_N (G_N(x, \pi) - (N-1)\delta_1)^2$$

$$= \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\}$$

$$+ \alpha_N ((N-1)\delta_1)^2 \text{ (since } G_N(x, \pi) = 0)$$

$$= \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\} + \alpha_N (N-1)(N-1)\delta_1^2.$$

Since $(N-1)\alpha_N > 1$, we obtain

$$\Phi_N(x, ((b_0, 0), (b_1, 0), \ldots, (b_{N-1}, 0)))$$

$$> \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1] c^2(b_n) + Var[h(b_n, Y_{n+1})]\} + (N-1)\delta_1^2 = \Phi_N(x, \pi).$$

The proof is completed.

Theorem 5. *Let* $x \in S$ *be an initial state and* A *be the target at the time horizon* N. *Assume that* $(N-1)\alpha_N > 1$. *Then there exists* $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U) - \mathcal{P}^*(N, U)$ *such that* $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ *and*

$$\Phi_N(x, \tilde{\pi}) < \Phi_N(x, ((b_0, 0), (b_1, 0), \ldots, (b_{N-1}, 0))).$$

Proof. Let $\pi = \{(b_n, \delta_n)\}_{n=0}^{N-1} \in \mathcal{P}^*(N, U)$ be such that $\delta_1 = \delta_2 = \cdots = \delta_{N-1} > 0$. From equation (11) and the iid property of $Z_1, Z_2, \ldots, Z_N$ (Assumption 1), we get

$$\Phi_N(x, \pi) = (N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\}$$

and $G_N(x, \pi) = 0.$

Choose a plan $\widetilde{\pi} = \{(\widetilde{b}_n, \widetilde{\delta}_n)\}_{n=0}^{N-1}$ defined by

$$\widetilde{\delta}_n = \frac{N-1}{N}\delta_n \quad \text{and} \quad \widetilde{b}_n = b_n, \; n = 0, 1, 2, \ldots, N-1.$$

Obviously, $\widetilde{\delta}_0 = 0$, $\widetilde{\delta}_1 = \widetilde{\delta}_2 = \cdots = \widetilde{\delta}_{N-1} > 0$ and

$$G_N(x, \widetilde{\pi}) = x - A + \sum_{n=1}^{N-1} \widetilde{\delta}_n + \sum_{n=0}^{N-1} (c(\widetilde{b}_n)E[Z_{n+1}] - E[h(\widetilde{b}_n, Y_{n+1})]) \neq 0.$$

Hence $\widetilde{\pi} \in \mathcal{P}(N, U) - \mathcal{P}^*(N, U)$. Next, we shall show that

$$\Phi_N(x, \widetilde{\pi}) < \Phi_N(x, ((\widetilde{b}_0, 0), (\widetilde{b}_1, 0), \ldots, (\widetilde{b}_{N-1}, 0))).$$

From equation (11) and the iid property of $Z_1, Z_2, \ldots, Z_N$, we get

$$\Phi_N(x, \widetilde{\pi})$$

$$= \sum_{n=1}^{N-1} \widetilde{\delta}_n^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(\widetilde{b}_n) + Var[h(\widetilde{b}_n, Y_{n+1})]\} + \alpha_N G_N^2(x, \widetilde{\pi})$$

$$= (N-1)\widetilde{\delta}_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(\widetilde{b}_n) + Var[h(\widetilde{b}_n, Y_{n+1})]\}$$

$$+ \alpha_N \left(x - A + (N-1)\widetilde{\delta}_1 + \sum_{n=0}^{N-1} \{E[Z_1]c(\widetilde{b}_n) - E[h(\widetilde{b}_n, Y_{n+1})]\} \right)^2$$

$$= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\}$$

$$+ \alpha_N \Bigg(x - A + (N-1)\frac{(N-1)}{N}\delta_1 + \frac{(N-1)}{N}\delta_1 - \frac{(N-1)}{N}\delta_1$$

$$+ \sum_{n=0}^{N-1} \{E[Z_1]c(b_n) - E[h(b_n, Y_{n+1})]\} \Bigg)^2$$

$$= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n)$$

$$+ Var[h(b_n, Y_{n+1})]\} + \alpha_N \left(G_N(x, \pi) - \frac{(N-1)}{N}\delta_1 \right)^2$$

$$= (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n)$$

$$+ Var[h(b_n, Y_{n+1})]\} + \alpha_N \left(\frac{(N-1)}{N}\delta_1 \right)^2$$

$$\leq (N-1)\frac{(N-1)^2}{N^2}\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n)$$

$$+ Var[h(b_n, Y_{n+1})]\} + \left(\frac{(N-1)}{N}\delta_1 \right)^2$$

$$= \left\{ \frac{(N-1)^2}{N^2} + \frac{N-1}{N^2} \right\}(N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\}$$

$$= \frac{N-1}{N}(N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\}$$

$$< (N-1)\delta_1^2 + \alpha_N \sum_{n=0}^{N-1} \{Var[Z_1]c^2(b_n) + Var[h(b_n, Y_{n+1})]\} = \Phi_N(x, \pi).$$

By virtue of Lemma 4, we have

$$\Phi(x, \pi) < \Phi_N(x, ((b_0, 0), (b_1, 0), \ldots, (b_{N-1}, 0)))$$

$$= \Phi_N(x, ((\tilde{b}_0, 0), (\tilde{b}_1, 0), \ldots, (\tilde{b}_{N-1}, 0))).$$

Thus

$$\Phi_N(x, \tilde{\pi}) < \Phi_N(x, ((\tilde{b}_0, 0), (\tilde{b}_1, 0), ..., (\tilde{b}_{N-1}, 0))).$$

The proof is now completed.

Lemma 6. *Let* $x \in S$ *be an initial state and* A *be the target at the time horizon* N. *Assume that* $(N-1)\alpha_N > 1$. *If* $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ *is an optimal plan, then* $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$.

Proof. Let $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ be an optimal plan. From equation (11), we have

$$\Phi_N(x, \tilde{\pi})$$

$$= \sum_{n=1}^{N-1} \tilde{\delta}_n^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} \{c^2(\tilde{b}_n) Var[Z_{n+1}] + Var[h(\tilde{b}_n, Y_{n+1})]\} + G_N^2(x, \tilde{\pi}) \right\}, \quad (15)$$

where

$$G_N(x, \tilde{\pi}) = x - A + \sum_{n=1}^{N-1} \tilde{\delta}_n + \sum_{n=0}^{N-1} \{c(\tilde{b}_n) E[Z_{n+1}] - E[h(\tilde{b}_n, Y_{n+1})]\}.$$

First, we shall show that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$. We work by a contradiction. Assume that $\tilde{\delta}_i \neq \tilde{\delta}_{i+1}$ for some $i \in \{1, 2, ..., N-2\}$. Let a plan $\pi_0 = \{(b_n, \delta_n)\}_{n=0}^{N-1}$ be defined by

$$\delta_n = \begin{cases} \dfrac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2}, & n = i, i+1 \\ \tilde{\delta}_n, & n \neq i, i+1 \end{cases} \quad \text{and} \quad b_n = \tilde{b}_n, \ n = 0, 1, 2, ..., N-1.$$

Obviously, $\pi_0 \in \mathcal{P}(N, U)$ and

$$G_N(x, \pi_0)$$

$$= x - A + \sum_{n=1}^{N-1} \delta_n + \sum_{n=0}^{N-1} c(b_n) E[Z_{n+1}] - \sum_{n=0}^{N-1} E[h(b_n, Y_{n+1})]$$

$$= x - A + \sum_{n=1; n \neq i, i+1}^{N-1} \delta_n + \delta_i + \delta_{i+1} + \sum_{n=0}^{N-1} c(b_n) E[Z_{n+1}] - \sum_{n=0}^{N-1} E[h(b_n, Y_{n+1})]$$

$$= x - A + \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \tilde{\delta}_n + 2\left(\frac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2}\right)$$

$$+ \sum_{n=0}^{N-1} c(\tilde{b}_n) E[Z_{n+1}] - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})]$$

$$= x - A + \sum_{n=1}^{N-1} \tilde{\delta}_n + \sum_{n=0}^{N-1} c(\tilde{b}_n) E[Z_{n+1}] - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})] = G_N(x, \tilde{\pi}). \tag{16}$$

Moreover, we have $\sum_{n=1}^{N-1} \delta_n^2 < \sum_{n=1}^{N-1} \tilde{\delta}_n^2$. To see this, we note that since $\tilde{\delta}_i \neq \tilde{\delta}_{i+1}$, $\tilde{\delta}_i$ and $\tilde{\delta}_{i+1}$ cannot be equal to zero at the same time. Hence $2\tilde{\delta}_i \tilde{\delta}_{i+1} < \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2$. Thus

$$\sum_{n=1}^{N-1} \delta_n^2 = \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \delta_n^2 + \delta_i^2 + \delta_{i+1}^2 = \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \tilde{\delta}_n^2 + 2\left(\frac{\tilde{\delta}_i + \tilde{\delta}_{i+1}}{2}\right)^2$$

$$= \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \tilde{\delta}_n^2 + \frac{1}{2}(\tilde{\delta}_i^2 + 2\tilde{\delta}_i \tilde{\delta}_{i+1} + \tilde{\delta}_{i+1}^2)$$

$$< \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \tilde{\delta}_n^2 + \frac{1}{2}(\tilde{\delta}_i^2 + \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2 + \tilde{\delta}_{i+1}^2)$$

$$= \sum_{n=1;\, n\neq i,\, i+1}^{N-1} \tilde{\delta}_n^2 + \tilde{\delta}_i^2 + \tilde{\delta}_{i+1}^2 = \sum_{n=1}^{N-1} \tilde{\delta}_n^2. \tag{17}$$

It follows from inequalities (15), (16) and (17) that $\Phi_N(x, \pi_0) < \Phi_N(x, \tilde{\pi})$ which is a contradiction. Hence $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$. Next, we try to show that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$. Assume that there exists $\tilde{\delta}_m = 0$ for some $m \in \{1, 2, ..., N-1\}$. Then $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} = 0$. By Theorem 5, there exists $\pi' = \{(b'_n, \delta'_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\delta'_1 = \delta'_2 = \cdots = \delta'_{N-1} > 0$ and $\Phi_N(x, \pi') < \Phi_N(x, \tilde{\pi})$. This contradicts the optimal plan of $\tilde{\pi}$ and then the proof is completed.

Next, we shall prove the existence of $\min_{\pi \in \mathcal{P}(N, U)} \Phi_N(x, \pi)$. We note that $\mathcal{P}(N, U)$ is a compact subset of the Euclidean space $(R^2)^N$. We can easily see this by utilizing the compactness property of $U = [\underline{b}, \bar{b}] \times [0, A]$ in R^2. What remains is to prove the continuity of $\Phi_N(x, \pi)$ on $\mathcal{P}(N, U)$.

Lemma 7. *The functions* $c(b)$ *and* $h(b, y)$ *are continuous on* $[\underline{b}, \bar{b}]$ *for each* y.

Proof. First, we will show that $h(b, y)$ is continuous on $[\underline{b}, \bar{b}]$ for each y.

Let $y \geq 0$ be fixed and $b_0 \in [\underline{b}, \bar{b}]$ be arbitrary. We shall prove by cases:

Case 1. $h(b, y) = by$. We get $\lim_{b \to b_0} h(b, y) = \lim_{b \to b_0} by = b_0 y = h(b_0, y)$.

Case 2. $h(b, y) = \min\{b, y\}$. We get

$$\lim_{b \to b_0} h(b, y) = \lim_{b \to b_0} \min\{b, y\} = \lim_{b \to b_0} \frac{1}{2}(b + y - |b - y|)$$

$$= \frac{1}{2}(b_0 + y - |b_0 - y|) = h(b_0, y).$$

From Cases 1 and 2, we have $h(b, y)$ is continuous on $[\underline{b}, \bar{b}]$.

Next, we will show that $c(b)$ is continuous on $[\underline{b}, \bar{b}]$. Note that

$$c(b) = c_0 - (1 + \theta_1)\frac{E[Y - h(b, Y)]}{E[Z]}$$

$$= c_0 - (1 + \theta_1)\frac{E[Y]}{E[Z]} + (1 + \theta_1)\frac{E[h(b, Y)]}{E[Z]}$$

$$= c_0 - (1 + \theta_1)\frac{E[Y]}{E[Z]} + \frac{1 + \theta_1}{E[Z]}\int_{\Omega} h(b, Y(\omega))dP(\omega)$$

$$= c_0 - (1 + \theta_1)\frac{E[Y]}{E[Z]} + \frac{1 + \theta_1}{E[Z]}\int_{-\infty}^{\infty} h(b, y) f_Y(y)dy, \tag{18}$$

where f_Y is the density function of Y.

Hence, it suffices to show that $\int_{-\infty}^{\infty} h(b, y) f_Y(y) dy$ is continuous on $[\underline{b}, \overline{b}]$. Let $b_0 \in [\underline{b}, \overline{b}]$ be arbitrary and $\tilde{g}$ be a function on R defined by

$$\tilde{g}(y) = y f_Y(y).$$

By Assumption 1, we have

$$\int_{-\infty}^{\infty} \tilde{g}(y) dy = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{\Omega} Y(\omega) dP(\omega) = E[Y] < \infty.$$

Since $h(b, y) \leq y$, $h(b, y) f_Y(y) \leq y f_Y(y) = \tilde{g}(y)$. By Lebesgue Dominated Convergence Theorem (Jones [5, p. 153]) and the continuity of $h(b, y)$ on $[\underline{b}, \overline{b}]$, we get

$$\lim_{b \to b_0} \int_{-\infty}^{\infty} h(b\ y) f_Y(y) dy = \int_{-\infty}^{\infty} \lim_{b \to b_0} h(b, y) f_Y(y) dy = \int_{-\infty}^{\infty} h(b_0, y) f_Y(y) dy.$$

Hence $\int_{-\infty}^{\infty} h(b, y) f_Y(y) dy$ is continuous on $[\underline{b}, \overline{b}]$ and then $c(b)$ is continuous on $[\underline{b}, \overline{b}]$.

Lemma 8. *The mapping* $F : U \mapsto R$ *defined by*

$$F(b, \delta) = \delta^2 + \alpha_N E\left[\frac{x - A}{N} + \delta + c(b) Z_1 - h(b, Y_1)\right]^2$$

is continuous on U.

Proof. By Assumption 1, the random variables Y_1 and Z_1 are independent, then we have

$$\begin{aligned}
F(b, \delta) &= \delta^2 + \alpha_N E\left[\frac{x - A}{N} + \delta + c(b) Z_1 - h(b, Y_1)\right]^2 \\
&= \int_{\Omega} \left(\delta^2 + \alpha_N \left[\frac{x - A}{N} + \delta + c(b) Z_1(\omega) - h(b, Y_1(\omega))\right]^2\right) dP(\omega) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\delta^2 + \alpha_N \left[\frac{x - A}{N} + \delta + c(b) z - h(b, y)\right]^2\right) f_{Y_1}(y) f_{Z_1}(z) dy dz \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f((y, z), (b, \delta)) dy dz, \qquad (19)
\end{aligned}$$

where

$$f((y, z), (b, \delta)) = \left(\delta^2 + \alpha_N\left[\frac{x-A}{N} + \delta + c(b)z - h(b, y)\right]^2\right) f_{Y_1}(y) f_{Z_1}(z)$$

and, f_{Y_1} and f_{Z_1} are the density functions of Y_1 and Z_1, respectively.

Let $\hat{g}$ be a function on R^2 defined by

$$\hat{g}(y, z) = (5A^2 + 2A(c_0 z + y) + c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z).$$

Now, we consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(y, z)\, dydz$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (5A^2 + 2A(c_0 z + y) + c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z)\, dydz$$

$$= 5A^2 + 2A\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (c_0 z + y) f_{Y_1}(y) f_{Z_1}(z)\, dydz$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z)\, dydz$$

$$= 5A^2 + 2A\int_{\Omega} (c_0 Z_1(\omega) + Y_1(\omega))\, dP(\omega) + \int_{\Omega} (c_0^2 Z_1^2(\omega) + Y_1^2(\omega))\, d(\omega)$$

$$= 5A^2 + 2A(c_0 E[Z_1] + E[Y_1]) + c_0^2 E[Z_1^2] + E[Y_1^2].$$

Since Y_1 and Z_1 have finite variances (Assumption 1), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{g}(y, z)\, dydz < \infty. \tag{20}$$

From equation (9) and inequality (19), we have

$$f((y, z), (b, \delta))$$

$$= \left(\delta^2 + \alpha_N\left[\frac{x-A}{N} + \delta + c(b)z - h(b, y)\right]^2\right) f_{Y_1}(y) f_{Z_1}(z)$$

$$\leq \left(\delta^2 + \left[\frac{x - A}{N} + \delta + c(b)z - h(b, y) \right]^2 \right) f_{Y_1}(y) f_{Z_1}(z)$$

$$= \left(\delta^2 + \left[\frac{x - A}{N} + \delta \right]^2 + 2\left[\frac{x - A}{N} + \delta \right] [c(b)z - h(b, y)] + [c(b)z - h(b, y)]^2 \right) f_{Y_1}(y) f_{Z_1}(z)$$

$$\leq \left(\delta^2 + \left[\frac{x - A}{N} + \delta \right]^2 + 2\left[\frac{x - A}{N} + \delta \right] [c(b)z + h(b, y)] + [c(b)z]^2 + [h(b, y)]^2 \right) f_{Y_1}(y) f_{Z_1}(z)$$

$$= (A^2 + 4A^2 + 2A[c_0 z + y] + c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z)$$

$$= (5A^2 + 2A[c_0 z + y] + c_0^2 z^2 + y^2) f_{Y_1}(y) f_{Z_1}(z) = \hat{g}(y, z). \tag{21}$$

By Lemma 7, for each fixed y and z, the function $f((y, z), (b, \delta))$ is continuous in the variable (b, δ) on U. So, by Lebesgue Dominated Convergence Theorem (Jones [5]), we obtain $F(b, \delta)$ is continuous on U.

Theorem 9. *Let $x \in S$ be fixed and A be the target at the time horizon N. Then $\Phi_N(x, \pi)$ is continuous on $\mathcal{P}(N, U)$.*

Proof. From equation (11) and the iid property of Y_n and Z_n, we have

$$\Phi_N(x, \pi)$$

$$= \sum_{n=1}^{N-1} \delta_n^2 + \alpha_N E\left[x - A + \sum_{n=1}^{N-1} \delta_n + \sum_{n=0}^{N-1} c(b_n) Z_{n+1} - \sum_{n=0}^{N-1} h(b_n, Y_{n+1}) \right]^2$$

$$= \sum_{n=0}^{N-1} \delta_n^2 + \alpha_N E\left[x - A + \sum_{n=0}^{N-1} \delta_n + \sum_{n=0}^{N-1} c(b_n) Z_{n+1} - \sum_{n=0}^{N-1} h(b_n, Y_{n+1}) \right]^2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\sum_{n=0}^{N-1} \delta_n^2 + \alpha_N \left[x - A + \sum_{n=0}^{N-1} \delta_n + \sum_{n=0}^{N-1} c(b_n) z_{n+1} \right.\right.$$

$$\left.\left. - \sum_{n=0}^{N-1} h(b_n, y_{n+1}) \right]^2 \right) f_{Y_1}(y_1) f_{Z_1}(z_1) \cdots f_{Y_1}(y_N) f_{Z_1}(z_N) dy_1 dz_1 \cdots dy_N dz_N$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f((y_1, \ldots, y_N, z_1, \ldots, z_N), \pi) dy_1 dz_1 \cdots dy_N dz_N,$$

where

$$f((y_1, \dots, y_N, z_1, \dots, z_N), \pi)$$

$$= \left(\sum_{n=0}^{N-1} \delta_n^2 + \alpha_N \left[x - A + \sum_{n=0}^{N-1} \delta_n + \sum_{n=0}^{N-1} c(b_n) z_{n+1} - \sum_{n=0}^{N-1} h(b_n, y_{n+1}) \right]^2 \right) f_{Y_1}(y_1) f_{Z_1}(z_1) \cdots f_{Y_1}(y_N) f_{Z_1}(z_N)$$

and, f_{Y_1} and f_{Z_1} are the density functions of Y_1 and Z_1, respectively.

Let $g*$ be a function on R^{2N} defined by

$$g*(y_1, \dots, y_N, z_1, \dots, z_N)$$

$$= \left((N + N^2) A^2 + 2NA \sum_{n=1}^{N} (c_0 z_n + y_n) + \sum_{n=1}^{N} (c_0^2 z_n^2 + y_n^2) + \sum_{m, n=1, n \neq m}^{N} (c_0^2 z_m z_n + y_m y_n) \right) f_{Y_1}(y_1) f_{Z_1}(z_1) \cdots f_{Y_1}(y_N) f_{Z_1}(z_N).$$

Since the sequences $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ are iid and have finite variances, we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g*(y_1, \dots, y_N, z_1, \dots, z_N) dy_1 dz_1 \cdots dy_N dz_N < \infty.$$

By the same proof as in inequality (21), we get

$$f((y_1, \dots, y_N, z_1, \dots, z_N), \pi) \leq g*(y_1, \dots, y_N, z_1, \dots, z_N).$$

Thus, by Lebesgue Dominated Convergence Theorem, we obtain Theorem 9.

Theorem 10. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$. Then there exists an optimal plan $\widetilde{\pi} = \{(\widetilde{b}_n, \widetilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\widetilde{\delta}_1 = \widetilde{\delta}_2 = \cdots = \widetilde{\delta}_{N-1} > 0$ and*

$$V_N(x) = (N-1)\widetilde{\delta}_1^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} Var[Z_1] c^2(\widetilde{b}_n) + \sum_{n=0}^{N-1} Var[h(\widetilde{b}_n, Y_{n+1})] + G_N^2(x, \widetilde{\pi}) \right\},$$

where

$$G_N(x, \tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + \sum_{n=0}^{N-1} E[Z_1]c(\tilde{b}_n) - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})].$$

Proof. From Theorem 9, we have $\Phi_N(x, \pi)$ is continuous on $\mathcal{P}(N, U)$. Since $\mathcal{P}(N, U)$ is a compact subset of $(R^2)^N$, there exists a plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\inf_{\pi \in \mathcal{P}(N, U)} \Phi_N(x, \pi) = \Phi_N(x, \tilde{\pi})$, i.e., $\tilde{\pi}$ is an optimal plan. By Lemma 6, we have $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$. From equations (7) and (11), we have

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ \sum_{n=0}^{N-1} Var[Z_1]c^2(\tilde{b}_n) + \sum_{n=0}^{N-1} Var[h(\tilde{b}_n, Y_{n+1})] + G_N^2(x, \tilde{\pi}) \right\},$$

where

$$G_N(x, \tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + \sum_{n=0}^{N-1} E[Z_1]c(\tilde{b}_n) - \sum_{n=0}^{N-1} E[h(\tilde{b}_n, Y_{n+1})].$$

This proves Theorem 10.

Corollary 11. *Let $x \in S$ be an initial state and A be the target at the time horizon N. Assume that $(N-1)\alpha_N > 1$ and $h(b, y)$ is the proportional reinsurance. Then there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\tilde{\pi}$ is stationary and*

$$V_N(x) = (N-1)\tilde{\delta}_1^2 + \alpha_N\{Nc^2(\tilde{b}_0)Var[Z_1] + N\tilde{b}_0^2 Var[Y_1] + G_N^2(x, \tilde{\pi})\}, \quad (22)$$

where $G_N(x, \tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + Nc(\tilde{b}_0)E[Z_1] - N\tilde{b}_0 E[Y_1]$.

Proof. From Theorem 10, there exists an optimal plan $\tilde{\pi} = \{(\tilde{b}_n, \tilde{\delta}_n)\}_{n=0}^{N-1} \in \mathcal{P}(N, U)$ such that $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1} > 0$ and

$$\Phi_N(x, \tilde{\pi}) = (N-1)\tilde{\delta}_1^2 + \alpha_N \left\{ Var[Z_1]\sum_{n=0}^{N-1} c^2(\tilde{b}_n) + Var[Y_1]\sum_{n=0}^{N-1} \tilde{b}_n^2 + G_N^2(x, \tilde{\pi}) \right\}, \quad (23)$$

where

$$G_N(x, \tilde{\pi}) = x - A + (N-1)\tilde{\delta}_1 + E[Z_1]\sum_{n=0}^{N-1} c(\tilde{b}_n) - E[Y_1]\sum_{n=0}^{N-1} \tilde{b}_n.$$

Next, we shall show that $\tilde{\pi}$ is a stationary plan. Since $\tilde{\delta}_1 = \tilde{\delta}_2 = \cdots = \tilde{\delta}_{N-1}$, we are left to show that $\tilde{b}_0 = \tilde{b}_1 = \cdots = \tilde{b}_{N-1}$. We work by a contradiction. Assume that $\tilde{b}_i \neq \tilde{b}_{i+1}$ for some $i \in \{1, 2, ..., N-2\}$. Let a plan $\pi_0 = \{(b_n, \delta_n)\}_{n=0}^{N-1}$ be defined by

$$b_n = \begin{cases} \dfrac{\tilde{b}_i + \tilde{b}_{i+1}}{2}, & n = i, i+1 \\ \tilde{b}_n, & n \neq i, i+1 \end{cases} \quad \text{and} \quad \delta_n = \tilde{\delta}_n \text{ for } n = 0, 1, 2, ..., N-1.$$

Obviously, $\pi_0 \in \mathcal{P}(N, U)$ and

$$G_N(x, \pi_0)$$

$$= x - A + (N-1)\delta_1 + E[Z_1]\sum_{n=0}^{N-1} c(b_n) - E[Y_1]\sum_{n=0}^{N-1} b_n$$

$$= x - A + (N-1)\delta_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i, i+1}^{N-1} c(b_n) + c(b_i) + c(b_{i+1})\right\} - E[Y_1]\left\{\sum_{n=0;\, n\neq i, i+1}^{N-1} b_n + b_i + b_{i+1}\right\}$$

$$= x - A + (N-1)\tilde{\delta}_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + 2c\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)\right\} - E[Y_1]\left\{\sum_{n=0;\, n\neq i, i+1}^{N-1} \tilde{b}_n + 2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)\right\}$$

$$= x - A + (N-1)\tilde{\delta}_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i, i+1}^{N-1} c(\tilde{b}_n) + 2\left\{c_0 - (1+\theta_1)\left(1 - \frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)\frac{E[Y_1]}{E[Z_1]}\right\}\right\} - E[Y_1]\sum_{n=0}^{N-1} \tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i,\, i+1}^{N-1} c(\tilde{b}_n) + \left\{2c_0 - (1+\theta_1)(2-\tilde{b}_i + \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right\}\right\} - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i,\, i+1}^{N-1} c(\tilde{b}_n) + \left\{2c_0 - (1+\theta_1)(1-\tilde{b}_i + 1 - \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right\}\right\} - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1$$

$$+ E[Z_1]\left\{\sum_{n=0;\, n\neq i,\, i+1}^{N-1} c(\tilde{b}_n) + \left\{2c_0 - (1+\theta_1)(1-\tilde{b}_i)\frac{E[Y_1]}{E[Z_1]} - (1+\theta_1)(1-\tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right\}\right\}$$

$$- E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1 + E[Z_1]\left\{\sum_{n=0;\, n\neq i,\, i+1}^{N-1} c(\tilde{b}_n) + c(\tilde{b}_i) + c(\tilde{b}_{i+1})\right\} - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n$$

$$= x - A + (N-1)\tilde{\delta}_1 + E[Z_1]\sum_{n=0}^{N-1} c(\tilde{b}_n) - E[Y_1]\sum_{n=0}^{N-1}\tilde{b}_n = G_N(x, \tilde{\pi}). \tag{24}$$

Moreover, we have $\sum_{n=0}^{N-1} b_n^2 < \sum_{n=0}^{N-1}\tilde{b}_n^2$ and $\sum_{n=0}^{N-1} c^2(b_n) < \sum_{n=0}^{N-1} c^2(\tilde{b}_n)$. To see this, we note that since $\tilde{b}_i \neq \tilde{b}_{i+1}$ (i.e., $\tilde{b}_i$ and $\tilde{b}_{i+1}$ cannot be equal to zero at the same time) and $c(\tilde{b}_i), c(\tilde{b}_{i+1}) > 0$, $2\tilde{b}_i\tilde{b}_{i+1} < \tilde{b}_i^2 + \tilde{b}_{i+1}^2$ and $2c(\tilde{b}_i)c(\tilde{b}_{i+1}) < c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1})$, respectively. This implies that

$$\sum_{n=0}^{N-1} b_n^2 = \sum_{n=0;\, n\neq i,\, i+1}^{N-1} b_n^2 + b_i^2 + b_{i+1}^2 = \sum_{n=0;\, n\neq i,\, i+1}^{N-1} \tilde{b}_n^2 + 2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)^2$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} \tilde{b}_n^2 + \frac{1}{2}(\tilde{b}_i^2 + 2\tilde{b}_i\tilde{b}_{i+1} + \tilde{b}_{i+1}^2)$$

$$< \sum_{n=0;\, n\neq i,\, i+1}^{N-1} \tilde{b}_n^2 + \frac{1}{2}\left(\tilde{b}_i^2 + \tilde{b}_i^2 + \tilde{b}_{i+1}^2 + \tilde{b}_{i+1}^2\right)$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} \tilde{b}_n^2 + \tilde{b}_i^2 + \tilde{b}_{i+1}^2 = \sum_{n=0}^{N-1} \tilde{b}_n^2 \qquad (25)$$

and

$$\sum_{n=0}^{N-1} c^2(b_n)$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(b_n) + c^2(b_i) + c^2(b_{i+1})$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + 2c^2\left(\frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + 2\left(c_0 - (1+\theta_1)\left(1 - \frac{\tilde{b}_i + \tilde{b}_{i+1}}{2}\right)\frac{E[Y_1]}{E[Z_1]}\right)^2$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(2c_0 - (1+\theta_1)(2 - \tilde{b}_i + \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(2c_0 - (1+\theta_1)(1 - \tilde{b}_i + 1 - \tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(2c_0 - (1+\theta_1)(1-\tilde{b}_i)\frac{E[Y_1]}{E[Z_1]} - (1+\theta_1)(1-\tilde{b}_{i+1})\frac{E[Y_1]}{E[Z_1]}\right)^2$$

$$= \sum_{n=0;\, n\neq i,\, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}\left(c(\tilde{b}_i) + c(\tilde{b}_{i+1})\right)^2$$

$$= \sum_{n=0; n \neq i, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}(c^2(\tilde{b}_i) + 2c(\tilde{b}_i)c(\tilde{b}_{i+1}) + c^2(\tilde{b}_{i+1}))$$

$$< \sum_{n=0; n \neq i, i+1}^{N-1} c^2(\tilde{b}_n) + \frac{1}{2}(c^2(\tilde{b}_i) + c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1}) + c^2(\tilde{b}_{i+1}))$$

$$= \sum_{n=0; n \neq i, i+1}^{N-1} c^2(\tilde{b}_n) + c^2(\tilde{b}_i) + c^2(\tilde{b}_{i+1}) = \sum_{n=0}^{N-1} c^2(\tilde{b}_n). \tag{26}$$

It follows from inequalities (23), (24), (25) and (26) that $\Phi_N(x, \pi_0) < \Phi_N(x, \tilde{\pi})$ which contradicts the optimal plan of $\tilde{\pi}$ and thus the proof is completed.

5. Example

In this section, we shall give an example of an optimal plan which can make a surplus approaching to the target A at a time horizon N. We begin by assuming that $h(b_0, y)$ is the proportional reinsurance with retention level b_0, an initial capital $x = 10$, a time horizon $N = 100$, the target $A = 60$ and $\alpha_N = 0.05,\ 0.1,\ 0.2$. We consider the safety loading of the insurer and reinsurer in three cases as follows: (a) $\theta_0 = 0.2, \theta_1 = 0.25$ (b) $\theta_0 = 0.25, \theta_1 = 0.25$ and (c) $\theta_0 = 0.3, \theta_1 = 0.25$. Suppose that the error of this estimate is $e = 0.1$. By Corollary 11, we know that the optimal plan π is stationary, thus it suffices to find b_0 and δ_1. We assume that $\{Y_n\}_{n=1}^{100}$ is a sequence of claims with iid exponential $Exp(1)$ and $\{Z_n\}_{n=1}^{100}$ is a sequence of inter-arrival times with iid Poisson $Poi(1)$. We solved for b_0, δ_1 under the conditions that $|E[X_N] - A| \leq 0.1$ (or equivalently, $|G_N(x, \pi)| \leq 0.1$) and $\Phi_N(x, \pi)$ is minimum. We get several optimal plans which satisfy the given parameters and the error $e = 0.1$.

Case (a). If $\theta_0 = 0.2$ and $\theta_1 = 0.25$, then the optimal plan is as follows:

$\pi = \{(b_0, \delta_0) = (0.74, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.74, 0.368)\}$ for $\alpha_N = 0.05$,

$\pi = \{(b_0, \delta_0) = (0.45, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.45, 0.441)\}$ for $\alpha_N = 0.1$,

$\pi = \{(b_0, \delta_0) = (0.26, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.26, 0.489)\}$ for $\alpha_N = 0.2$.

Case (b). If $\theta_0 = 0.25$ and $\theta_1 = 0.25$, then the optimal plan is as follows:

$$\pi = \{(b_0, \delta_0) = (0.65, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.65, 0.340)\} \text{ for } \alpha_N = 0.05,$$

$$\pi = \{(b_0, \delta_0) = (0.42, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.42, 0.398)\} \text{ for } \alpha_N = 0.1,$$

$$\pi = \{(b_0, \delta_0) = (0.23, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.23, 0.446)\} \text{ for } \alpha_N = 0.2.$$

Case (c). If $\theta_0 = 0.3$ and $\theta_1 = 0.25$, then the optimal plan is as follows:

$$\pi = \{(b_0, \delta_0) = (0.57, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.57, 0.310)\} \text{ for } \alpha_N = 0.05,$$

$$\pi = \{(b_0, \delta_0) = (0.34, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.34, 0.368)\} \text{ for } \alpha_N = 0.1,$$

$$\pi = \{(b_0, \delta_0) = (0.17, 0), (b_1, \delta_1) = \cdots = (b_{99}, \delta_{99}) = (0.17, 0.411)\} \text{ for } \alpha_N = 0.2.$$

Note that, $V_N(x)$ can be calculated by putting b_0, δ_1 and $c(b_0)$, $(c(b_0) := \theta_0 - \theta_1 + (1 + \theta_1)b_0)$ into equation (22). The values of these parameters for each case are shown in Table 1:

Table 1. The values of b_0, δ_1 and V_N for each case

	Case (a)	Case (b)	Case (c)
	$\theta_0 = 0.2, \theta_1 = 0.25$	$\theta_0 = 0.25, \theta_1 = 0.25$	$\theta_0 = 0.30, \theta_1 = 0.25$
	b_0 : δ_1 : V_N	b_0 : δ_1 : V_N	b_0 : δ_1 : V_N
$\alpha_N = 0.05$	0.74 : 0.368 : 19.9733	0.65 : 0.340 : 16.8581	0.57 : 0.310 : 14.0456
$\alpha_N = 0.1$	0.45 : 0.441 : 23.9060	0.42 : 0.398 : 20.0232	0.34 : 0.368 : 16.8197
$\alpha_N = 0.2$	0.26 : 0.489 : 26.5391	0.23 : 0.446 : 22.4057	0.17 : 0.411 : 18.8600

Finally, suppose that the time horizon N and the safety loading of the reinsurer θ_1 are fixed. By virtue of Remark 3, for each fixed retention level, we can find a shareholder input which satisfies the condition that $|E[X_N] - A| \le e$ for a given error e. Hence, if the error of estimate is decreased, then we can still find an optimal plan according to Corollary 11.

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