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A VALUE FUNCTION OF DISCRETE-TIME SURPLUS PROCESS IN INSURANCE UNDER INVESTMENT AND REINSURANCE CREDIT RISK

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Abstract

This paper has studied an insurance model where the surplus process can be controlled by two activities, one is reinsurance for which the reinsurance company has an opportunity to default and other is an investment in a financial market. We prove the existence of an optimal plan and derive a formula for the value function which is the minimum of total discounted cost function in the framework of discrete-time surplus process.

1. Introduction

In recent years, risk models have attracted much attention in the insurance business, in connection with the possible insolvency and the capital reserves of the insurance company. The main interest from the point of view of an insurance

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company is claim arrival and claim size, which affect the capital of the company. Claims happen at the times T_i, satisfying $0 = T_0 \le T_1 \le T_2 \le \cdots$. We call them *claim arrivals* or, simply, *arrivals*. The nth claim arriving at time causes the claim size Y_n. The *inter-arrival times* or, simply, *inter-arrival*, $Z_n := T_n - T_{n-1}$ is the length of time between $(n-1)$th claim and nth claim. By period n, we shall mean the random interval $[T_{n-1}, T_n)$, $n \ge 1$.

Now let the constant c_0 represent the premium rate for one unit time; the random variable $c_0 \sum_{i=1}^{n+1} Z_i = c_0 T_{n+1}$ describes the inflow of capital into the business by time T_{n+1}, and $\sum_{i=1}^{n+1} Y_i$ describes the outflow of capital due to payments for claims occurring in $[0, T_{n+1}]$. Therefore, the quantity

$$X_{n+1} = x + c_0 T_{n+1} - \sum_{i=1}^{n+1} Y_i \tag{1.1}$$

is the insurer's balance (or surplus) at time T_{n+1} with the constant $X_0 = x > 0$ as the initial capital. In summary, the discrete-time surplus process will be defined as follows:

$$X_0 = x, \quad X_{n+1} = X_n + c_0 Z_{n+1} - Y_{n+1}, \quad n = 0, 1, 2, \dots. \tag{1.2}$$

Reinsurance and investment are a normally activity of insurance company because reinsurance can reduce the risk (ruin probability) arising from claims, and the investment can make more profit for the company. Thus there are many papers which studied their effect in the insurance business. For example, the effect of reinsurance on ruin probability was studied by Dickson and Waters [3], minimizing ruin probability in a continuous-time model considered by Browne [2], Hipp and Plum [4], Hipp and Vogt [5], Højgaard and Taksar [6, 7], Schmidli [9], and exponential utility and minimizing ruin probability in a discrete-time model considered by Schäl [8].

In this paper, we shall prove the existence of an optimal plan (i.e., the strategy or policy of choosing reinsurance and investment for minimizing a value function) and derive a formula of the value function under the condition that a reinsurer has opportunity to default and an investment in risky assets. Let $\{X_n, n \ge 1\}$ be the surplus process which can be controlled by choosing the retention level b of reinsurance for one period, and for each level b, the insurer has to pay the premium rate to the reinsurer which is deducted from c_0, as a result of which the insurer's income rate will be represented by the function $c(b)$. The level $\overline{b}$ stands for the control action without reinsurance, so that $c_0 = c(\overline{b})$ and the level $\underline{b}$ is the smallest retention level which can be chosen. Of course, we obtain the *net income rate* $c(b)$, where $0 \leq c(b) \leq c_0$, for all $b \in [\underline{b}, \overline{b}]$ and $c(b)$ is increasing. By the *expected value principle*, $c(b)$ can be calculated as follows:

$$c(b) = c_0 - (1 + \theta_0) \cdot \frac{E[Y - h(b, Y)]}{E[Z]}, \tag{1.3}$$

where θ_0 is the safety loading of the reinsurer and the function $h(b, y)$ is the part of the claim size y paid by the insurer, and the remaining part $y - h(b, y)$ which called *reinsurance recovery* is paid by the reinsurer.

Next, we shall recall the *reinsurance credit risk* (RCR) which is the risk of the reinsurance counterparty failing to pay reinsurance recoveries in full to the ceding company (insurer) in a timely manner, i.e., unwillingness to pay, or even not paying them at all. Therefore, we assume that for each retention level $b \in [\underline{b}, \overline{b}]$, the reinsurer has an opportunity to default, i.e., the insurer has to pay

$$\begin{cases} y, & \text{if reinsurer default with probability } P(K = 0) = p, \\ h(b, y), & \text{if reinsurer dose not default with probability } P(K = 1) = 1 - p, \end{cases}$$

where K is a random variable with value in $\{0, 1\}$ and $p \in [0, 1)$ is constant. The random variable K is said to be *binary recovery*. Let $\{T_n, n \geq 0\}$ be a sequence of arrival and let K_n be a binary recovery at time T_n.

In addition, the insurer can invest the surplus (capital) in a financial market with m risky assets, called *stocks*, described by the price process

$$\{S_n = (S_n^1, S_n^2, \ldots, S_n^m), n \geq 1\},$$

where $S_n^k > 0$ is the price of one share of stock k at the time T_{n-1}. We now define the return process $\{R_n = (R_n^1, R_n^2, \ldots, R_n^m), n \geq 1\}$ by $R_n^k = (S_n^k - S_{n-1}^k)/S_{n-1}^k, 1 \leq k \leq m$. For each $n \geq 1$, a portfolio vector $\delta_{n-1} = (\delta_{n-1}^1, \delta_{n-1}^2, \ldots, \delta_{n-1}^m) \in \mathbb{R}^m$ specifies the time T_{n-1} and the component δ_{n-1}^k represents the amount invested in stock k during period n. This means that the insurance company holds $\delta_{n-1}^k / S_{n-1}^k$ shares of stock k during period n, so that the value of these shares at the time T_n is $\delta_{n-1}^k \cdot S_n^k / S_{n-1}^k$.

In this situation, we will allow for a negative value for δ_{n-1}^k, that is, we admit the short selling of stocks. Letting X_n be a surplus and (b_n, δ_n) be a control action at the time T_n, therefore, we can adapt the surplus process (1.2) as follows:

$$
\begin{aligned}
X_{n+1} &= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1 - K_{n+1}) - \sum_{k=1}^{m} \delta_n^k + \sum_{k=1}^{m} \frac{\delta_n^k}{S_n^k} S_{n+1}^k \\
&= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1 - K_{n+1}) + \sum_{k=1}^{m} \delta_n^k \frac{(S_{n+1}^k - S_n^k)}{S_n^k} \\
&= X_n + c(b_n)Z_{n+1} - h(b_n, Y_{n+1})K_{n+1} - Y_{n+1}(1 - K_{n+1}) + \sum_{k=1}^{m} \delta_n^k R_{n+1}^k \\
&= X_n + c(b_n)Z_{n+1} - \{h(b_n, Y_{n+1})K_{n+1} + Y_{n+1}(1 - K_{n+1})\} + \langle \delta_n, R_{n+1} \rangle,
\end{aligned}
$$

where $X_0 = x$ and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^m$. It is convenient to set

$$
X_0 = x, \quad X_{n+1} = X_n + L(b_n, \delta_n, K_{n+1}, R_{n+1}, Y_{n+1}, Z_{n+1}), \quad n = 0, 1, 2, \ldots, \tag{1.4}
$$

where

$$
L(b, \delta, k, r, y, z) = c(b) \cdot z - \{h(b, y)k + y(1 - k)\} + \langle \delta, r \rangle. \tag{1.5}
$$

If we let $f(x, b, \delta, k, r, y, z) = x + L(b, \delta, r, y, z)$, then f is the *system function* as mentioned in Bersekas and Shreve [1]. We see that the process $\{X_n, n \geq 0\}$ is driven by the control action $u_n = (b_n, \delta_n)$ and the sequence of random vector $\{W_n, n \geq 1\}$, where $W_n = (K_n, R_n, Y_n, Z_n)$ (the disturbance for period n) is the source of the randomness of the model. It is natural to assume that the process W_n is iid, i.e., we make the following assumption:

Assumption 1. Independence assumption (IA)

$W_n = (K_n, R_n, Y_n, Z_n)$, $n \geq 1$ are independent and identically distributed random variables (iid). In addition, it is assumed that (K_n, Y_n, Z_n) and R_n are independent for all $n \geq 1$.

As a consequence, $\{R_n, n \geq 1\}$ as well as $\{K_n, n \geq 1\}$, $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ are also iid. We set $K = K_1$, $R = R_1$, $Y = Y_1$, $Z = Z_1$ and $W = W_1$ for a typical binary recovery, typical return, typical period length, typical claim, and typical disturbance, respectively.

2. Dynamic Programming with Finite Horizon

In this section, we assume that all the processes are defined in a probability space $(\Omega, \mathcal{F}, P)$. Let $\{X_n, n \geq 0\}$ be a surplus process (as in Section 1) with value in a state space $(S, \mathcal{S})$ which is a measurable space. Suppose that $\{X_n, n \geq 0\}$ is driven by a sequence of iid random variables $\{W_n, n \geq 1\}$ with values in a measurable space $(E, \mathcal{E})$. Therefore, $(E, \mathcal{E})$ is called *disturbance space*. The surplus process can be controlled at the beginning of every period in a measurable space $(U, \mathcal{U})$ which is called the *control action space*. In addition, the model is specified by the following quantities:

- $\alpha \in [0, 1]$ is the *discount factor*;
- $g : S \times U \to (-\infty, \infty]$ is the *one-period cost function*, which is measurable and bounded from below;
- $N \in \{1, 2, 3, \dots\}$ is a *time horizon* (number of periods) and
- $\hat{V}_N : S \to (-\infty, \infty]$ is the *terminal cost function* for time horizon N, which is measurable and bounded from below.

Definition 2.1. A *plan* for the time horizon N over action space U is a (finite) sequence $\pi = (u_i)_{i=0}^{N-1}$ of control action $u_i \in U$, for all $i \in \{0, 1, 2, \dots, N-1\}$. A set of all plans for the time horizon N over action space U is denoted $\mathcal{P}(N, U)$. A plan $\pi \in \mathcal{P}(N, U)$ is said to be *u-stationary*, if $\pi = \underbrace{(u, u, \dots, u)}_{N \text{ terms}}$ for some $u \in U$.

For each initial state $x \in S$ and plan $\pi = (u_i)_{i=0}^{N-1}$, the surplus process (1.4) can be written by

$$X_{n+1} = X_n + L(u_n, W_{n+1}) = x + \sum_{k=0}^{n} L(u_k, W_{k+1}), \quad n = 0, 1, 2, \dots, N-1 \quad (2.1)$$

and $X_0 = x$.

For the state $X_n = x_n$, the cost at the time T_n will be $g(x_n, u_n)$ and the next state

$$x_{n+1} = x_n + c(b_n)z_n - (h(b_n, y_{n+1})k_{n+1} + y_{n+1}(1 - k_{n+1})) + \langle \delta_n, r_{n+1} \rangle \tag{2.2}$$

which will result in a cost $g(x_{n+1}, u_{n+1})$ at the time T_{n+1}. Thus, the present value of the costs at the time T_{n+1} will be $\alpha \cdot g(x_{n+1}, u_{n+1})$, i.e., $g(x_{n+1}, u_{n+1})$ is discounted by α.

Definition 2.2. Let N be the time horizon. Then the *total discounted cost function* and the *valued function for the time horizon N* are defined by

$$\Phi^{(N)}(x, \pi) = E\left[\sum_{i=0}^{N-1} \alpha^i g(X_i, u_i) + \alpha^N \hat{V}_N(X_N) \mid X_0 = x\right], \quad \text{where} \quad \pi = (u_i)_{i=0}^{N-1}, \tag{2.3}$$

and

$$V^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N, U)} \Phi^{(N)}(x, \pi), \quad \text{respectively.} \tag{2.4}$$

A plan $\pi \in \mathcal{P}(N, U)$ is said to be *optimal*, if $V^{(N)}(x) = \Phi^{(N)}(x, \pi)$. If π is u-stationary, then we write $\Phi^{(N)}(x, \pi) = \Phi^{(N)}(x, u)$.

3. Main Results

In this section, we study the insurance model introduced in Section 1 under the assumption that the insurer can borrow an unlimited amount of money. Let the state space $S = \mathbb{R}$ and the control space $U = [\underline{b}, \overline{b}] \times \mathbb{R}^m$. Thus, for each state $x \in S$, we can choose any control actions $u = (b, \delta) \in [\underline{b}, \overline{b}] \times \mathbb{R}^m$, where b is the retention level of reinsurance and $\delta = (\delta^1, \delta^2, \ldots, \delta^m)$ is the portfolio vector.

We will study the cost structure which is given by the idea that the insurance company is not insolvency (ruined) but only penalized if the size of the surplus is negative or small. The penalty cost of being in state x is of the form $\text{const} \times e^{-\beta x}$ for some $\beta > 0$. Therefore, we define the cost functions as

$$g(x, u) = \gamma \cdot e^{-\beta x}, \quad \hat{V}_N(x) = v_0 \cdot e^{-\beta x}, \quad \text{for some } \gamma, v_0 \geq 0, \tag{3.1}$$

when $x \in S$, $u \in U$. Thus, we obtain the total discounted cost function of model (1.4) as

$$\Phi^{(N)}(x, \pi) = E\left[\sum_{i=0}^{N-1} \alpha^i \gamma \cdot e^{-\beta x_i} + \alpha^N \nu_0 \cdot e^{-\beta X_N} \mid X_0 = x\right], \tag{3.2}$$

where $\pi \in \mathcal{P}(N, U)$.

In this paper, we will use the method of dynamic programming to prove the main theorem. In order to do this, we define $\Phi_n^{(N)}(x, \pi)$ and $V_n^{(N)}(x)$ as follows:

$$\Phi_n^{(N)}(x, \pi) = E\left[\sum_{i=n}^{N-1} \alpha^{i-n} \gamma \cdot e^{-\beta X_i} + \alpha^{N-n} \nu_0 \cdot e^{-\beta X_N} \mid X_n = x\right], \quad n = 0, 1, 2, \dots, N-1 \tag{3.3}$$

$$\Phi_N^{(N)}(x, \pi) = \nu_0 \cdot e^{-\beta x}, \quad \text{where} \quad \pi = (u_i)_{i=0}^{N-1}, \tag{3.4}$$

and

$$V_n^{(N)}(x) = \inf_{\pi \in \mathcal{P}(N, U)} \Phi_n^{(N)}(x, \pi), \quad n = 0, 1, 2, \dots, N-1, \tag{3.5}$$

$$V_N^{(N)}(x) = \Phi_N^{(N)}(x, \pi). \tag{3.6}$$

It is obvious to see that $\Phi^{(N)}(x) = \Phi_0^{(N)}(x)$ and $V^{(N)}(x) = V_0^{(N)}(x)$. For each $\pi = (u_0, u_1, u_2, \dots, u_{N-1}) \in \mathcal{P}(N, U)$, we can see from equation (3.3) that

$$\Phi_n(x, \pi) = \Phi_n(x, (u_0, u_1, u_2, \dots, u_{n-1}, u_n, \dots, u_{N-1}))$$

does not depend on the control actions $u_0, u_1, \dots, u_{n-1}$. Therefore, (3.5) becomes

$$V_n^{(N)}(x) = \inf_{u_n, u_{n+1}, \dots, u_{N-1} \in U} \Phi_n^{(N)}(x, (u_0, u_1, u_2, \dots, u_{n-1}, u_n, \dots, u_{N-1})). \tag{3.7}$$

Next, we define a function $G : U \to [0, \infty]$ by

$$G(u) := E[e^{-\beta L(u, W_1)}], \tag{3.8}$$

for all $u \in U$, where W_1 is given in Assumption 1 (IA). Thus, by Assumption 1 (IA), we have

$$E[e^{-\beta L(u, W_n)}] = E[e^{-\beta L(u, W_1)}], \tag{3.9}$$

for all $u \in U$ and $n \in \{1, 2, 3, \dots\}$.

Remark 3.1. By Assumption 1 (IA), for each $\pi = (u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$, (3.3) becomes

$$\Phi_n^{(N)}(x, \pi) = \gamma e^{-\beta x} + \alpha G(u_n)\Phi_{n+1}^{(N)}(x, \pi), \tag{3.10}$$

for all $n = 0, 1, 2, \ldots, N-1$.

Proof of Remark 3.1. Let $\pi = (u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$. In the case of $n = N-1$, we have

$$\begin{aligned} \Phi_{N-1}^{(N)}(x, \pi) &= E[\gamma e^{-\beta x} + \alpha v_0 \cdot e^{-\beta(x+L(u_{N-1}, W_N))}] \\ &= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_{N-1}, W_N)}] v_0 \cdot e^{-\beta x} \\ &= \gamma e^{-\beta x} + \alpha G(u_{N-1})\Phi_N^{(N)}(x, \pi). \end{aligned} \tag{3.11}$$

In the case of $0 \le n < N-1$. Consider

$$\Phi_n^{(N)}(x, \pi)$$

$$= E\left[\sum_{i=n}^{N-1} \alpha^{i-n}\gamma e^{-\beta X_i} + \alpha^{N-n} v_0 \cdot e^{-\beta X_N} \mid X_n = x\right]$$

$$= E\left[\gamma e^{-\beta x} + \alpha\gamma e^{-\beta(x+L(u_n, W_{n+1}))} + \sum_{i=n+2}^{N-1} \alpha^{i-n}\gamma e^{-\beta\left(x+\sum_{j=n}^{i-1} L(u_j, W_{j+1})\right)} + v_0\alpha^{N-n} e^{-\beta\left(x+\sum_{j=n}^{N-1} L(u_j, W_{j+1})\right)}\right]$$

$$= \gamma e^{-\beta x} + \alpha E\left[e^{-\beta L(u_n, W_{n+1})}\left\{\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma\alpha^{i-(n+1)}\gamma e^{-\beta\left(x+\sum_{j=n+1}^{i-1} L(u_j, W_{j+1})\right)} + v_0\alpha^{N-(n+1)} e^{-\beta\left(x+\sum_{j=n+1}^{N-1} L(u_j, W_{j+1})\right)}\right\}\right]. \tag{3.12}$$

Since the $\{W_{n+1},\, n \geq 0\}$ is an independent sequence, $\{L(u_n,\, W_{n+1}),\, n \geq 0\}$ is also an independent sequence. Thus, we obtain

$$\Phi_n^{(N)}(x,\, \pi)$$

$$= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_n, W_{n+1})}] E\left[\gamma e^{-\beta x} + \sum_{i=n+2}^{N-1} \gamma \alpha^{i-(n+1)} e^{-\beta\left(x + \sum_{j=n+1}^{i-1} L(u_j,\, W_{j+1})\right)} \right.$$

$$\left. + \nu_0 \alpha^{N-(n+1)} e^{-\beta\left(x + \sum_{j=n+1}^{N-1} L(u_j,\, W_{j+1})\right)} \right]$$

$$= \gamma e^{-\beta x} + \alpha E[e^{-\beta L(u_n, W_{n+1})}] E\left[\sum_{i=n+1}^{N-1} \alpha^{i-(n+1)} \gamma e^{-\beta X_i} + \alpha^{N-n} \nu_0 e^{-\beta X_N} \mid X_{n+1} = x \right]$$

$$= \gamma e^{-\beta x} + \alpha G(u_n) \Phi_{n+1}^{(N)}(x,\, \pi). \tag{3.13}$$

This proves Remark 3.1. □

Remark 3.1 leads to the following lemma:

Lemma 3.2. *Under Assumption* 1, *let* $x \in S$ *be an initial state and* $u \in U$ *be a control action. If* $G(u) < \infty$, *then*

$$\Phi_n^{(N)}(x,\, u) = \begin{cases} (\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n}) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u)}, & \alpha G(u) \neq 1, \\ (\gamma(N - n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u) = 1, \end{cases}$$

for all $n = 0,\, 1,\, 2,\, \ldots,\, N$.

Proof. In the case of $\alpha G(u) \neq 1$, we will prove this lemma by using mathematical induction. Obviously, the case $n = N$ holds. Now assume that

$$\Phi_{n+1}^{(N)}(x,\, u) = (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u))](\alpha G(u))^{N-(n+1)}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}, \tag{3.14}$$

where $n + 1 < N$. By virtue of Remark 3.1, we get

$$\Phi_n^{(N)}(x, u)$$

$$= \gamma e^{-\beta x} + \alpha G(u)\Phi_{n+1}^{(N)}(x, u)$$

$$= \gamma e^{-\beta x} + \alpha G(u)(\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-(n+1)}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}$$

$$= (\gamma(1 - \alpha G(u)) + (\gamma\alpha G(u) - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n})) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}$$

$$= (\gamma - [\gamma - \nu_0(1 - \alpha G(u))](\alpha G(u))^{N-n}) \cdot \frac{e^{-\beta x}}{1 - \alpha G(u)}. \quad (3.15)$$

Obviously, the case $\alpha G(u) = 1$ holds. This proves Lemma 3.2. □

Lemma 3.3. *Under Assumption 1, let $x \in S$ be an initial state. If there exists $u^* \in U$ such that $G(u^*) = \min_{u \in U} E[e^{-\beta L(u, W)}] < \infty$, then*

$$V_n^{(N)}(x) = \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x) \quad (3.16)$$

and the u^*-stationary is an optimal plan, i.e., $V^{(N)}(x) = \Phi^{(N)}(x, u^*)$.

Proof. Let $n \in \{0, 1, 2, \dots, N-1\}$. Then, by equation (3.7) and Remark 3.1, we have

$$V_n^{(N)}(x) = \inf_{u_n, u_{n+1}, \dots, u_{N-1} \in U} \Phi_n^{(N)}(x, (u_0, u_1, u_2, \dots, u_{N-1})) \quad (3.17)$$

$$= \gamma e^{-\beta x} + \alpha \inf_{u_n, u_{n+1}, \dots, u_{N-1} \in U} \{G(u_n)\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \dots, u_{N-1}))\}. \quad (3.18)$$

For each $(u_i)_{i=0}^{N-1} \in \mathcal{P}(N, U)$, we have $\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \dots, u_{N-1})) \geq 0$ and $G(u_n) \geq 0$, for all $n \in \{0, 1, 2, \dots, N-1\}$, and $\Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \dots, u_{N-1}))$ does not depend on the control actions $u_0, u_1, \dots, u_n$. Therefore, (3.18) becomes

$$V_n^{(N)}(x) = \gamma e^{-\beta x} + \alpha \inf_{u_n \in U} G(u_n) \cdot \inf_{u_{n+1}, \dots, u_{N-1} \in U} \Phi_{n+1}^{(N)}(x, (u_0, u_1, u_2, \dots, u_{N-1})), \quad (3.19)$$

$$= \gamma e^{-\beta x} + \alpha G(u^*) \cdot \inf_{\pi \in \mathcal{P}(N, U)} \Phi_{n+1}^{(N)}(x, \pi) \quad (3.20)$$

$$= \gamma e^{-\beta x} + \alpha G(u^*) \cdot V_{n+1}^{(N)}(x). \quad (3.21)$$

From Remark 3.1 and (3.21), since $V_N^{(N)}(x) = \Phi_N^{(N)}(x, u^*)$, we conclude that

$$V_n^{(N)}(x) = \Phi_n^{(N)}(x, u^*), \tag{3.22}$$

for all $n \in \{0, 1, 2, 3, ..., N-1\}$. Therefore, $V^{(N)}(x) = \Phi^{(N)}(x, u^*)$. This means that u^*-stationary is an optimal plan. □

From Lemma 3.3, we need the condition for the existence of $\min_{u \in U} G(u)$ which can be shown by using the extreme value theorem. First, we need the property that $u \to G(u)$ is continuous, so we make the following assumption:

Assumption 2. Continuity assumption (CA)

The functions $c(b)$ and $h(b, y)$ are continuous in b (for each y) and

$$E[e^{\beta \cdot Y}] < \infty, \quad E[e^{\varepsilon \cdot \| R \|}] < \infty, \quad \text{for all} \quad \varepsilon > 0. \tag{3.23}$$

Since (K, Z, Y) and R are assumed to be independent as in Assumption 1 (IA) and since $0 \le h(b, Y)K + Y(1-K) \le YK + Y(1-K) = Y$,

$$E[e^{-\beta(c(b)Z - \{h(b,Y)K + Y(1-K)\} + \langle \delta, R \rangle)}] \le E[e^{\beta\{h(b,Y)K + Y(1-K)\} - \beta \langle \delta, R \rangle}]$$

$$\le E[e^{\beta\{h(b,Y)K + Y(1-K)\}}] E[e^{-\beta \langle \delta, R \rangle}]$$

$$\le E[e^{\beta Y}] \cdot E[e^{|\beta| \cdot \| \delta \| \cdot \| R \|}] < \infty.$$

We now conclude that $G(u)$ is continuous by using the dominated convergence theorem. Moreover, $b \mapsto E[e^{-\beta(c(b)Z - \{h(b,Y)K + Y(1-K)\})}]$ and $\delta \mapsto [e^{-\beta \langle \delta, R \rangle}]$ are also continuous.

Recall that, we have already set $K = K_1$, $R = R_1$, $Z = Z_1$ and $Y = Y_1$ but sometimes, in Assumption 3, we write K_1, R_1, Z_1 and Y_1 instead of K, R, Z and Y, respectively, to emphasize the balance's surplus at time T_1.

Assumption 3. No-arbitrage assumption (NA)

For any portfolio vector $\delta \in \mathbb{R}^m$, $P(\langle \delta, R \rangle \ge 0) = 1$, implies $P(\langle \delta, R \rangle = 0) = 1$.

In the investment, the investor will look for the *arbitrage opportunity*, i.e., they want to hold the portfolio $\delta_0 \in \mathbb{R}^m$ such that $P(\langle \delta_0, R_1 \rangle \geq 0) = 1$, which implies that for the initial surplus $X_0 = x$, we have

$$\begin{aligned} X_1 &= x + c(b_0)Z_1 - \{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\} + \langle \delta_0, R_1 \rangle \\ &\geq x + c(b_0)Z_1 - \{h(b_0, Y_1)K_1 + Y_1(1 - K_1)\} \quad \text{a.s.} \end{aligned} \tag{3.24}$$

which means that the portfolio $\delta_0 \in \mathbb{R}^m$ has no risk. Of course, the investor would like to use this opportunity because the quantity $P(\langle \delta_0, R_1 \rangle > 0)$ may be positive which indicates an arbitrage opportunity. Note that Assumption 3 (NA) is equivalent to

"for any portfolio $\delta \in \mathbb{R}^m$, $0 < P(\langle \delta, R \rangle < 0) < 1$ or $\langle \delta, R \rangle = 0$ a.s." (NA*)

By using (NA*), we have

$$\mathbb{R}^m = \mathfrak{I} \cup \mathfrak{I}^* \quad \text{and} \quad \mathfrak{I} \cup \mathfrak{I}^* \neq \varnothing,$$

where $\mathfrak{I} = \{\delta \in \mathbb{R}^m : \langle \delta, R \rangle = 0 \text{ a.s.}\}$ and $\mathfrak{I}^* = \{\delta \in \mathbb{R}^m : 0 < P(\langle \delta, R \rangle < 0) < 1\}$. It is easy to see that $\mathfrak{I}$ is a linear subspace of $\mathbb{R}^m$. Thus, there exists a linear subspace $\mathfrak{I}^\perp$ of $\mathbb{R}^m$ such that $\mathbb{R}^m = \mathfrak{I} \oplus \mathfrak{I}^\perp$ and $\mathfrak{I} \cap \mathfrak{I}^\perp = \{0\}$ ($\mathbb{R}^m$ is the direct sum of $\mathfrak{I}$ and $\mathfrak{I}^\perp$) which implies $\mathfrak{I}^\perp \backslash \{0\} \subset \mathfrak{I}^*$.

Lemma 3.4. *Under Assumptions* 1-3, *let* $\delta \in \mathbb{R}^m$ *be given. If* $\delta \in \mathfrak{I}^\perp \backslash \{0\}$, *then there exists an* $\varepsilon > 0$ *such that* $E[-\langle \delta, R \rangle 1_{(\langle \delta, R \rangle < 0)}] \geq \varepsilon \cdot P(\langle \delta, R \rangle \leq -\varepsilon) > 0$.

Proof. Let $\delta \in \mathfrak{I}^\perp \backslash \{0\}$. Then, by (NA*), we have $P(\langle \delta, R \rangle < 0) := q$ for some $q > 0$. Let $A_n := \{\delta \in \Omega : \langle \delta, R \rangle \leq -1/n\}$ and $A_\infty := \{\delta \in \Omega : \langle \delta, R \rangle < 0\}$. Obviously, $A_n \subset A_{n+1} \subset A_\infty$, for all $n = 1, 2, 3, \ldots$ and $\bigcup_{n=1}^{\infty} A_n = A_\infty$. Thus, $\{P(A_n)\}_{n=1}^{\infty}$ is an increasing sequence and then

$$\lim_{l \to \infty} P(A_l) = \lim_{l \to \infty} P\left(\bigcup_{n=1}^{l} A_n\right) = P(A_\infty) = q.$$

So that there exists $l_0 \in \mathbb{N}$ such that $P(A_{l_0}) > q/2$, i.e., $P(\langle \delta, R\rangle \leq -1/l_0) > q/2$. By Markov's inequality, we have

$$\begin{aligned} l_0 E[-\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)}] &\geq P(-\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)} \geq 1/l_0) \\ &= P(\langle \delta, R\rangle 1_{(\langle \delta, R\rangle < 0)} \leq -1/l_0) \\ &= P(\langle \delta, R\rangle \leq -1/l_0) \\ &> q_0/2 > 0. \end{aligned}$$

Choose $\varepsilon = 1/l_0$. The lemma follows. □

Theorem 3.1. *Under Assumptions* 1-3, *let* $x \in S$ *be an initial state. Then there exists* $u^* = (b^*, \delta^*) \in U$ *such that*

$$G(u^*) = \min_{(b, \delta) \in U} E[e^{-\beta(c(b) Z - \{h(b, Y) K + Y(1-K)\} + \langle \delta, R\rangle)}] < \infty$$

and

$$V^{(N)}(x) = \begin{cases} (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u^*))](\alpha G(u^*))^N) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \neq 1, \\ (\gamma(N-n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

Moreover, u^**-stationary is an optimal plan.*

Proof. By Assumption 1 (IA), we have

$$\inf_{u \in U} G(u) = \inf_{b \in [\underline{b}, \overline{b}]} E[e^{-\beta(c(b) Z - \{h(b, Y) K + Y(1-K)\})}] \inf_{\delta \in \mathbb{R}^m} E[e^{-\beta\langle \delta, R\rangle}].$$

Since $[\underline{b}, \overline{b}]$ is compact and $b \mapsto E[e^{-\beta(c(b) Z - \{h(b, Y) K + Y(1-K)\})}]$ is continuous, by using extreme value theorem, there exists $b^* \in [\underline{b}, \overline{b}]$ such that

$$E[e^{-\beta(c(b^*) Z - \{h(b^*, Y) K + Y(1-K)\})}] = \min_{b \in [\underline{b}, \overline{b}]} E[e^{-\beta(c(b) Z - \{h(b, Y) K + Y(1-K)\})}]. \quad (3.25)$$

Next, we will find the minimizer of $E[e^{-\beta\langle \delta, R\rangle}]$ over $\mathbb{R}^m$. We consider the following cases:

Case 1. If $\Im = \mathbb{R}^m$, then by (NA*), we can see that $E[e^{-\beta\langle\delta, R\rangle}] = 1$, for all $\delta \in \mathbb{R}^m$.

Case 2. If $\Im \subset \mathbb{R}^m$, then $\Im^{\perp} \neq \{0\}$. By using Lemma 3.4, we can show that for each $\delta \in \Im^{\perp}\backslash\{0\}$, there exists $\varepsilon > 0$ such that

$$E[-\langle\delta, R\rangle 1_{(\langle\delta, R\rangle<0)}] \geq \varepsilon \cdot P(\langle\delta, R\rangle \leq -\varepsilon) > 0.$$

Hence

$$\begin{aligned}
\lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle}] &= \lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle} 1_{(\langle\delta, R\rangle<0)} + e^{-\beta\langle n\cdot\delta, R\rangle} 1_{(\langle\delta, R\rangle\geq 0)}] \\
&\geq \lim_{n\to\infty} E[e^{-\beta\langle n\cdot\delta, R\rangle} 1_{(\langle\delta, R\rangle<0)}] \\
&\geq \lim_{n\to\infty} e^{\beta n E[-\langle\delta, R\rangle 1_{(\langle\delta, R\rangle<0)}]} \\
&\geq \lim_{n\to\infty} e^{\beta n \varepsilon P(\langle\delta, R\rangle<-\varepsilon)} \\
&= \infty. \qquad (3.26)
\end{aligned}$$

Next, for each $\kappa > 0$, we define $F_\kappa := \{\delta \in \Im^{\perp} : \|\delta\| = 1, E[e^{-\beta\langle\kappa\cdot\delta, R\rangle}] \leq 2\}$. Let κ_1 and κ_2 be two real numbers such that $\kappa_2 > \kappa_1 > 0$. If $F_{\kappa_2} \neq \varnothing$, then

$$\begin{aligned}
E[e^{-\beta\langle\kappa_1\cdot\delta, R\rangle}] &= E\left[e^{-\beta\frac{\kappa_1}{\kappa_2}\cdot\kappa_2\langle\delta, R\rangle + \frac{\kappa_2-\kappa_1}{\kappa_2}\cdot 0}\right] \\
&\leq \frac{\kappa_1}{\kappa_2} E[e^{-\beta\kappa_2\langle\delta, R\rangle}] + \frac{\kappa_2 - \kappa_1}{\kappa_2} \\
&\leq \frac{2\kappa_1}{\kappa_2} + \frac{\kappa_2 - \kappa_1}{\kappa_2} = \frac{\kappa_2 + \kappa_1}{\kappa_2} < 2, \quad \text{for all } \delta \in F_{\kappa_2}.
\end{aligned}$$

This means that $F_{\kappa_1} \supset F_{\kappa_2}$, for all $\kappa_2 > \kappa_1 > 0$. Since F_κ is a compact for all $\kappa > 0$, and by inequality (3.26), we have $\bigcap_{n\in\mathbb{N}} F_n = \varnothing$. Thus, by the nested sequence property, there exists an $n_0 \in \mathbb{N}$ such that $F_n = \varnothing$, for all $n \geq n_0$. This implies that $F_\kappa = \varnothing$, for all $\kappa \geq n_0$ and this is equivalent to $\partial B_\kappa := \{\delta \in \Im^{\perp} : \|\delta\| = \kappa, E[e^{-\beta\langle\delta, R\rangle}] \leq 2\} = \varnothing$, for all $\kappa \geq n_0$. Therefore, we have

$$\inf_{\delta\in\mathbb{R}^m} E[e^{-\beta\langle\delta, R\rangle}] = \inf_{\delta\in\mathbb{R}^m} E[e^{-\beta(\langle\rho(\delta), R\rangle+\langle\delta-\rho(\delta), R\rangle)}]$$

$$= \inf_{\delta\in\mathbb{R}^m} E[e^{-\beta\langle\rho(\delta), R\rangle}]$$

$$= \inf_{\delta\in\Im^{\perp}} E[e^{-\beta\langle\delta, R\rangle}]$$

$$= \inf_{\delta\in\Im^{\perp}, \|\delta\|\le n_0} E[e^{-\beta\langle\delta, R\rangle}], \tag{3.27}$$

where $\rho: \mathbb{R}^m \to \mathbb{R}^m$ is an orthogonal projection on $\Im^{\perp}$. Since $\{\delta \in \Im^{\perp} : \|\delta\| \le n_0\}$ is compact and $\delta \mapsto E[e^{-\beta\langle\delta, R\rangle}]$ is continuous, there exists $\delta^* \in \{\delta \in \Im^{\perp} : \|\delta\| \le n_0\}$ such that

$$E[e^{-\beta\langle\delta^*, R\rangle}] = \inf_{\delta\in\Im^{\perp}, \|\delta\|\le n_0} E[e^{-\beta\langle\delta, R\rangle}]. \tag{3.28}$$

Therefore, $u^* = (b^*, \delta^*)$ is a minimizer of $G(u)$. By Lemma 3.4, we see that u^*-stationary is an optimal plan. Also, from Lemma 3.2, we obtain

$$V^{(N)}(x) = \begin{cases} (\gamma - [\gamma - \nu_0 \cdot (1 - \alpha G(u^*))](\alpha G(u^*))^N) \cdot \dfrac{e^{-\beta x}}{1 - \alpha G(u^*)}, & \alpha G(u^*) \ne 1, \\ (\gamma(N - n) + \nu_0) \cdot e^{-\beta x}, & \alpha G(u^*) = 1. \end{cases}$$

This completes the proof.

References

[1] D. Bersekas and S. E. Shreve, Stochastic Optimal Control; the Discrete-time Case, Academic Press, New York, 1978.

[2] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, Math. Oper. Res. 20 (1995), 937-958.

[3] D. C. M. Dickson and H. R. Waters, Reinsurance and ruin, Insur. Math. Econom. 19 (1996), 61-80.

[4] C. Hipp and M. Plum, Optimal investment for insurers, Insur. Math. Econom. 27 (2000), 215-228.

[5] C. Hipp and M. Vogt, Optimal XL Insurance, Technical Report, Univers. Karlsruhe, 2001.

[6] B. Højgaard and M. Taksar, Optimal proportional reinsurance policies for diffusion models, Scand. Actuar. J. 2 (1998), 166-180.

[7] B. Højgaard and M. Taksar, Optimal proportional reinsurance policies for diffusion models with transaction costs, Insur. Math. Econom. 22(1) (1998), 41-51.

[8] M. Schäl, On discrete-time dynamic programming in insurance: exponential utility and maximizing the ruin probability, Scand. Actuarial J. 3 (2004), 189-210.

[9] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, Scand. Actuarial J. (2001), 55-68.