STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume LIII, Number 4, December 2008

ON THE FRACTIONAL STOCHASTIC FILTERING

TRAN HUNG THAO, PAIROTE SATTAYATHAM, AND TIDARUT PLIENPANICH

Abstract. The aim of this note is to introduce an approximate approach to fractional filtering problems, where either the signal process or observation process, or both are perturbed by a fractional noise. Approximate filtering equations are established and the true filtering is considered as the limit case of approximate filterings.

1. Introduction

It is known that fractional Brownian motion (fBm) was introduced firstly by B. Mandelbrot and Van Nees. This is a centered Gaussian process $B^H = \{B_t^H,\ t \geq 0\}$ with covariance

$$E(B_s^H B_t^H) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big) \tag{1.1}$$

where H is called the Hurst parameter, $0 < H < 1$

In the case where $H = \frac{1}{2}$, $E(B_s^{1/2} B_t^{1/2}) = \frac{1}{2}(s+t-|t-s|)$ we have an ordinary standard Brownian motion. The fractional Brownian motion is in general neither a martingale nor a Markov process. In contrary, it exhibits a long-range dependence. Some approaches to fractional stochastic calculus have been introduced by L. Coutin, L. Decreusefond, W. Dai, C. Heyde, Lin, A.S. Üstünel, D. Feyel, de La Pradelle, T. Duncan, B. Duncan (refer for example to [1, 2, 3])

A fractional Brownian motion has been considered also by C. Ciesielski and al. as a special sequence of random functions in some Orlicz-Besov space [4].

Received by the editors: 02.02.2007.
2000 *Mathematics Subject Classification.* 78M35, 60K40.
Key words and phrases. Liouville fractional Brownian motion, approximation by semimartingales, filtering.

Stochastic filtering problems in fractional stochastic framework were studied by some authors. The main obstacle in the study of these problems is the fact that the signal process or the observation process is not driven by a martingale and powerful tools of martingale theory can not be applied as in traditional stochastic filtering theory. Some attempts have been made by L. Decreusefond, A.A. Üstünel to overcome this difficulty by using the Malliavin Calculus [3].

In a fractional filtering problem, the state (or signal) process is some stochastic process X_t, while the observation Y_t is given by a fractional process of the form:

$$Y_t = \int_0^t h_s ds + B_t^H \tag{1.2}$$

where B_t^H is a fractional Brownian motion with a Hurst parameter H such that $0 < H < \frac{1}{2}$ and $h_t = h(X_t)$ is some process of finite energy, i.e.

$$E \int_0^\infty h_s^2 ds < \infty.$$

As shown in [6], the fBm $B^H = (B_t^H,\ t \geq 0)$ has the following representation

$$B_t^H = \frac{1}{\Gamma(1-\alpha)} \left\{ Z_t + \int_0^t (t-s)^\alpha dW_s \right\}, \tag{1.3}$$

where $\{W_s,\ s \in \mathbb{R}\}$ is a standard Brownian motion, $\alpha = H - \frac{1}{2} \in \big(-\frac{1}{2}, \frac{1}{2}\big)$ and $Z_t = \int_{-\infty}^0 \big[(t-s)^\alpha - (-s)^\alpha\big] dW_s$. Since the process Z_t has absolutely continuous trajectories, it suffices to consider the term

$$B_t = \int_0^t (t-s)^\alpha dW_s. \tag{1.4}$$

In fact, B_t is a fractional Brownian motion of the Liouville form.

In our filtering problems, we consider the observation process Y_t of the form

$$Y_t = \int_0^t h_s ds + B_t\,, \tag{1.5}$$

with B_t defined by (1.4), where h_t is some continuous process, $h_t = h(X_t)$.

Since $B_t = \int_0^t (t-s)^\alpha dW_s$ can be approximated by a semimartingale B_t^ε as shown below, our filtering problem can be considered as the limit case of following filtering problems when $\varepsilon \to 0$:

$$\begin{aligned} &\text{Signal process:} && X_t\ , \\ &\text{Observation process:} && Y_t^\varepsilon = \int_0^t h_s ds + B_t^\varepsilon, \end{aligned} \tag{1.6}$$

where B_t^ε is some semimartingale for every $\varepsilon > 0$.

2. L^2-approximation for B_t

Let B_t^H be the fractional noise in the observation process Y_t in (1.5) and W_t the corresponding Brownian motion in its representation (1.3). Suppose that $0 < \alpha < \frac{1}{2}$, where $\alpha = H - \frac{1}{2}$.

Define

$$B_t = \int_0^t (t-s)^\alpha dW_s\,, \tag{2.1}$$

and

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha dW_s\,, \tag{2.2}$$

for every $\varepsilon > 0$.

The Ito stochastic differential of B_t^ε is then:

$$dB_t^\varepsilon = \Big(\int_0^t \alpha(t-s+\varepsilon)^{\alpha-1} dW_s\Big)dt + \varepsilon^\alpha dW_t\,. \tag{2.3}$$

Indeed by applying of the stochastic theorem of Fubini, we have

$$\begin{aligned} \int_0^t\int_0^s (s-u+\varepsilon)^{\alpha-1} dW_s ds &= \int_0^t [\int_u^s (s-u+\varepsilon)^{\alpha-1} ds] dW_u \\ &= \frac{1}{\alpha}[\int_0^t (t-u+\varepsilon)^\alpha dW_u - \varepsilon^\alpha W_t] \\ &= \frac{1}{\alpha}[B_t^\varepsilon - \varepsilon^\alpha W_t]. \end{aligned}$$

Therefore

$$B_t^\varepsilon = \alpha\int_0^t [\int_0^s (s-u+\varepsilon)^{\alpha-1} dW_s] dt + \varepsilon^\alpha W_t\,,$$

or

$$B_t^\varepsilon = \int_0^t \alpha\varphi_s^\varepsilon ds + \varepsilon^\alpha W_t\,, \tag{2.4}$$

or equivalently,

$$dB_t^\varepsilon = \alpha\varphi_t^\varepsilon ds + \varepsilon^\alpha dW_t\,, \tag{2.5}$$

where

$$\varphi_t^\varepsilon = \int_0^t (s-u+\varepsilon)^{\alpha-1} dW_s \,,$$

so B_t^ε is a semimartingale.

We would like to recall here a fundamental result [8] on the L^2- convergence of semimartingales B_t^ε to the fractional process B_t as $\varepsilon \to 0$. Basing on this result we introduce an approximate approach to fractional filtering problems.

Theorem 2.1. *B_t^ε converges to B_t in $L^2(\Omega)$ when ε tends to 0. This convergence is uniform with respect to $t \in [0,T]$.*

3. Fractional filtering for a general signal process

We consider first a filtering problem where the signal process is a general stochastic process $(X_t, t \geq 0)$ with $E|X_t| < \infty$ for every $t > 0$ and the observation process Y_t is given by

$$Y_t = \int_0^t h_s ds + B_t \;,\; 0 \leq t \leq T, \tag{3.1}$$

where B_t is the fractional process given by (1.4) and $h_t = h(X_t)$ is a continuous process with

$$E\int_0^t h_s^2 ds < \infty \,. \tag{3.2}$$

Now for every $\varepsilon > 0$ we establish a new filtering problem (an 'approximate' one), where the signal process is $(X_t, 0 \leq t \leq T)$, $E|X_t|^2 < \infty$ and the observation process is

$$Y_t^\varepsilon = \int_0^t h_s ds + B_t^\varepsilon \;,\; 0 \leq t \leq T \tag{3.3}$$

where B_t^ε is given by (2.2), and T is some positive real number.

From now on, we take $\varepsilon = \frac{1}{n}$ and put

$\mathcal{F}_t = \mathcal{F}_t^Y$: σ-algebra generated by $(Y_s, s \leq t)$

$\mathcal{F}_t^{(n)} = \mathcal{F}_t^{Y^{1/n}}$: σ-algebra generated by $(Y_s^{1/n}, s \leq t)$.

Define the filter π_t of (X_t) based on observations (Y_t) as the following conditional expectation

$\pi_t(X) = E(X|\mathcal{F}_t)$, or more general

$\pi_t(f) = E(f(X)|\mathcal{F}_t), f$ is any continuous and bounded function on $\mathbb{R} : f \in C_b(\mathbb{R})$.

Denote also by $\pi_t^{(n)}$ the filter of X based on observation $Y_t^{1/n}$:

$$\pi_t^{(n)}(X) = E(X|\mathcal{F}_t^{Y^{1/n}})$$

and

$$\pi_t^{(n)}(f) = E(f(X)|\mathcal{F}_t^{Y^{1/n}}), f \in C_b(\mathbb{R}).$$

Theorem 3.1. *The filter $\pi_t^{(n)}$ converges to π_t in $L^2(\Omega, \mathcal{F}, P)$ as $n \to \infty$.*

Proof. Consider two observations

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{1/n} \tag{3.4}$$

and

$$Z_t^{1/n} = \int_0^t h_s ds + B_{t+\frac{1}{n}}\ , \tag{3.5}$$

where $B_t^{1/n} = \int_0^t (t + \frac{1}{n} - s)^\alpha dW_s$ and $B_{t+\frac{1}{n}} = \int_0^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^\alpha dW_s$.

We observe that

$$E|Y_t^{1/n} - Z_t^{1/n}|^2 = E|B_t^{1/n} - B_{t+\frac{1}{n}}|^2 = E|\int_t^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^\alpha dW_s|^2$$

$$= \int_t^{t+\frac{1}{n}} (t + \frac{1}{n} - s)^{2\alpha} ds = \frac{1}{2\alpha+1}\frac{1}{n^{2\alpha+1}} \to 0 \ \ (n \to \infty) \tag{3.6}$$

where the last equality of (3.6) holds by virtue of the Itô isometry.

Now it follows from the convergence

$$\|Y_t^{1/n} - Z_t^{1/n}\|_{L^2} \to 0 \text{ as } n \to \infty$$

that

$$\|E(X_t|Y_t^{1/n}) - E(X_t|Z_t^{1/n})\|_{L^2} \to 0 \text{ as } n \to \infty \text{ (refer to [5])}$$

or more general

$$\|E(X_t|\mathcal{F}_t^{Y^{1/n}}) - E(X_t|\mathcal{F}_t^{Z^{1/n}})\|_{L^2} \;\to\; 0 \text{ as } n \to \infty\,,$$

and for $f \in C_b(\mathbb{R})$:

$$\|E\big(f(X_t)|\mathcal{F}_t^{Y^{1/n}}\big) - E\big(f(X_t)|\mathcal{F}_t^{Z^{1/n}}\big)\|_{L^2} \;\to\; 0 \text{ as } n \to \infty\,. \tag{3.7}$$

Since the family of σ-algebras $\mathcal{F}_t^{Z^{1/n}}$ is non-increasing such that $\bigcap\limits_n \mathcal{F}_t^{Z^{1/n}} = \mathcal{F}_t^Y$ then it follows from a Levy theorem on the convergence of conditional expectations that (refer to [5] or to [7]):

$$E\big(f(X_t)|\mathcal{F}_t^{Z^{1/n}}\big) \xrightarrow{L^2} E\big(f(X_t)|\mathcal{F}_t^{Y}\big) \;\to\; 0 \text{ as } n \to \infty\,. \tag{3.8}$$

By combining (3.7) and (3.8) and using the Minkowski inequality we have

$$E\big(f(X_t)|\mathcal{F}_t^{Y^{1/n}}\big) \xrightarrow{L^2} E\big(f(X_t)|\mathcal{F}_t^{Y}\big) \;\to\; 0 \text{ as } n \to \infty\,, \tag{3.9}$$

or $\pi_t^{(n)} \to \pi_t$ in L^2 as $n \to \infty$ by notation of filters. □

4. **Fractional filtering for a semimartingale**

In this section we consider a filtering problem where the signal process is a semimartingale

$$X_t = X_0 + \int_0^t H_s ds + V_t\,, \tag{4.1}$$

where V_t is a Brownian motion and H_t is a stochastic process such that

$$E\int_0^t H_s^2 ds < \infty\,, \tag{4.2}$$

and the observation is the fractional process

$$Y_t = \int_0^t h(X_s)ds + B_t\,, \tag{4.3}$$

where B_t is a fractional Brownian motion defined as in (1.4) such that the corresponding Brownian motion W_t in this expression is independent of V_t, and that

$$E\int_0^t h^2(X_s)ds < \infty\,. \tag{4.4}$$

As in Section II we can consider the 'approximate' filtering problem:

Signal process:

$$X_t = X_0 + \int_0^t H_s ds + V_t \,. \tag{4.5}$$

Observation process:

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{1/n} \quad , \tag{4.6}$$

where $B_t^{1/n}$ is given by (2.2), and $h_t = h(X_t)$.

Replacing B_t^{ε} in (4.6) for $\varepsilon = \frac{1}{n}$ by its expression in (2.4) we have:

$$Y_t^{1/n} = \int_0^t h_s ds + \alpha \int_0^t \varphi_s^{1/n} ds + \frac{1}{n^\alpha} W_t, \; 0 \le t \le T \;, \tag{4.7}$$

where $\varphi_t^{1/n} = \displaystyle\int_0^t (t - s + \frac{1}{n})^{\alpha-1} dW_s \,.$

Put $\bar{h}_s = h_s + \alpha \varphi_s^{1/n}$, then (4.7) becomes:

$$Y_t^{1/n} = \int_0^t \bar{h}_s ds + \frac{1}{n^\alpha} W_t \;, 0 \le t \le T \;, \tag{4.8}$$

So $Y_t^{1/n}$ is a $\mathcal{F}_t^W$- semimartingale. Notice that

$$\bar{h}_s^2 \le 2(h_s^2 + \alpha^2 (\varphi_s^{1/n})^2) \,,$$

$$E\bar{h}_s^2 \le 2Eh_s^2 + \alpha^2 E(\varphi_s^{1/n})^2 \,,$$

But by the Ito isometry, we see that

$$E(\varphi_s^{1/n})^2 = E[(\int_0^t (t - s + \frac{1}{n})^{\alpha-1} dW_s)^2]$$

$$= \int_0^t (t - s + \frac{1}{n})^{2\alpha-2} ds \le \int_0^T (t - s + \frac{1}{n})^{2\alpha-2} ds < \infty.$$

It follows from Fubini's theorem that

$$E \int_0^t \bar{h}_s^2 ds < \infty \tag{4.9}$$

Now define the innovation process:

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h}) ds \tag{4.10}$$

then $\nu_t^{1/n}$ is a $\mathcal{F}_t^{Y^{1/n}}$- martingale.

Now we are in the position to write down the FKK (Fujisaki - Kallianpur - Kunita) equation for the filtering problem (4.1)- (4.3):

$$\pi_t^{(n)}(f) = \pi_0^{(n)}(f) + \int_0^t \pi_s^{(n)}(f(H))ds + \int_0^t [\pi_s^{(n)}(f(X)\bar{h}) - \pi_s^{(n)}(f(X))\pi_s^{(n)}(\bar{h})]d\nu_s^{1/n} \quad , \tag{4.11}$$

where $f \in C_b(\mathbb{R})$ and $\pi_0^{(n)}(f) = E(f(X_0)|\mathcal{F}_0^{Y^{1/n}})$.

Notice that from (4.5) we have

$$\begin{aligned} E|X_t| &\leq E|X_0| + E|\int_0^t H_s ds| + E|V_t| \\ &\leq E|X_0| + T.E\int_0^t H_s^2 ds < \infty. \end{aligned}$$

then by the Levy theorem we can see that $L^2\text{-}\lim_{n\to\infty} \pi_t^{(n)}$ exists as in proof of Theorem 3.1. Now we can state:

Theorem 4.1. *The filter $\pi_t(f(X)) = E(f(X_t)|\mathcal{F}_t^Y) = L^2\text{-}\lim_{n\to\infty} \pi_t^{(n)}(f)$ exists, where $\pi_t^{(n)}(f)$ satisfies the equation (4.11).*

5. **General fractional filtering**

Suppose now that the signal process $(X_t,\ 0 \leq t \leq T)$ and the observation process $(Y_t,\ 0 \leq t \leq T)$ are fractional processes given by

$$X_t = X_0 + \int_0^t H_s ds + B_t^{(1)} \quad , \ E|X_t| < \infty, \tag{5.1}$$

$$Y_t = \int_0^t h_s ds + B_t^{(2)} \quad , \tag{5.2}$$

where

$$B_t^{(1)} = \int_0^t (t-s)^\beta dU_s \quad , \tag{5.3}$$

$$B_t^{(2)} = \int_0^t (t-s)^\alpha dW_s \quad , \tag{5.4}$$

U_s and W_t are two independent standard Brownian motions, $\beta = H_1 - \frac{1}{2},\ \alpha = H_2 - \frac{1}{2}$, H_1 and H_2 are two Hurst parameters and $0 < \alpha,\ \beta < \frac{1}{2}$.

H_t and h_t are $\mathcal{F}_t$-adapted process, $h_t = h(X_t)$ with continuous function $h(.)$, such that

$$E\int_0^t H_s^2 ds < \infty \quad , \tag{5.5}$$

$$E\int_0^t h_s^2 ds < \infty \quad , \tag{5.6}$$

As before we consider an "approximate model" for the filtering problem (5.1)-(5.6) as follows:

Signal:

$$X_t^{1/n} = X_0 + \int_0^t H_s ds + B_t^{(1)1/n},\ 0 \le t \le T \quad , \tag{5.7}$$

Observation:

$$Y_t^{1/n} = \int_0^t h_s ds + B_t^{(2)1/n},\ 0 \le t \le T \quad , \tag{5.8}$$

where $h_t = h(X_t^{1/n})$ and

$$B_t^{(1)1/n} = \int_0^t (t - s + \frac{1}{n})^{\beta} dU_s$$

and

$$B_t^{(2)1/n} = \int_0^t (t - s + \frac{1}{n})^{\alpha} dW_s\,. \tag{5.9}$$

The filter $\pi_t^{(n)}$ for the problem (5.7)-(5.8) is defined as

$$\pi_t^{(n)}(f) = E[f(X_t^{1/n})|\mathcal{F}_t^{Y_t^{1/n}}], \quad f \in C_b(\mathbb{R}) \tag{5.10}$$

and we will verify if the filter π_t for the original problem (5.1)-(5.6) can be defined as a L^2-limit of $\pi_t^{(n)}$ as $n \to \infty$.

We need the following lemma (refer to [7]).

Lemma 5.1. *Let (X_n) be a sequence of random variables such that for every n, $|X_n| \le Y$, where Y is integrable. If $(\mathcal{F}_n)$ is an increasing (resp. decreasing) sequence of σ-algebras, then $E[X_n|\mathcal{F}_n]$ converges a.s to $E[X|\mathcal{F}]$, where $\mathcal{F} = \sigma(\cup\mathcal{F}_n)$ (resp. $\mathcal{F} = \cap\mathcal{F}_n$)*

Proof. Take $\varepsilon > 0$ and put

$$A = \inf_{k \geq m} X_n, \; B = \sup_{k \geq m} X_n \tag{5.11}$$

where m is chosen such that

$$E[B - A] < \varepsilon \quad . \tag{5.12}$$

Then for $n \geq m$ we have

$$E[A|\mathcal{F}_n] \leq E[X_n|\mathcal{F}_n] \leq E[B|\mathcal{F}_n]. \tag{5.13}$$

The left and right- hand sides of (5.13) are martingales converging a.s. to $E(A|\mathcal{F})$ and $E(B|\mathcal{F})$ respectively. We have

$$E(A|\mathcal{F}) \leq \underline{lim}\ E(X_n|\mathcal{F}_n) \leq \overline{lim}\ E(X_n|\mathcal{F}_n) \leq E(B|\mathcal{F}) \quad , \tag{5.14}$$

and

$$E(A|\mathcal{F}) \leq E(X|\mathcal{F}) \leq E(B|\mathcal{F}) \quad , \tag{5.15}$$

It follows that

$$E[\underline{lim}\ E(X_n|\mathcal{F}_n) - \overline{lim}\ E(X_n|\mathcal{F}_n)] \leq \varepsilon \quad ,$$

hence $E(X_n|\mathcal{F}_n)$ converges a.s. and the limit is $E(X|\mathcal{F})$.

Remark. The Lemma 5.1 still holds if we replace the a.s. convergence by the L^2- convergence (refer to [5]).

Theorem 5.1. *Under the conditions given by (5.1)-(5.8) the filter* $\pi_t(f) = E[f(X_t)|\mathcal{F}_t^Y]$ *is determined by*

$$\pi_t(f) = L^2\text{-}\lim\ \pi_t^{(n)}(f), \; f \in C_b(\mathbb{R})$$

where $\pi_t^{(n)}$ *satisfies the following filtering equation*

$$\pi_t^{(n)}(f) = \pi_0^{(n)}(f) + \int_0^t \pi_s^{(n)}(f(\bar{H}))ds + \int_0^t [\pi_s^{(n)}(f(X)\bar{h}) - \pi_s^{(n)}(f(X))\pi_s^{(n)}(\bar{h})]d\nu_s^{1/n} \quad , \tag{5.16}$$

and

$$\bar{H}_t = H_t + \beta\psi_t^{1/n} \text{ , where } \psi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\beta dU_t \, , \tag{5.17}$$

$$\bar{h}_t = h_t + \alpha\varphi_t^{1/n} \quad \text{, where } \; \varphi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\alpha dW_t \, , \tag{5.18}$$

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h})ds, \tag{5.19}$$

H_t and h_t satisfy conditions (5.5) and (5.6) and $X_t^{1/n}$ and $Y_t^{1/n}$ are defined by (5.7), (5.8) and (5.9) for $0 \leq t \leq T$.

Proof. It follows from the definition (5.7) for the process $X_t^{1/n}$ and from Theorem 2.1 that $X_t^{1/n} \to X_t$ in $L^2(\Omega, \mathcal{F}, P)$ as $n \to \infty$.

As for $Y_t^{1/n}$ defined by (5.8) we can see that

$$Y_t^{1/n} - Y_t = \int_0^t [h(X_s^{1/n}) - h(X_s)]ds + B_t^{(2)1/n} - B_t^{(2)}\,, \tag{5.20}$$

where $h : R \to R$ is a continuous function by assumption, then the L^2-convergence of $B_t^{(2)1/n}$ and $X_t^{1/n}$ respectively to $B_t^{(2)}$ and X_t implies that of $Y_t^{1/n}$ to Y_t.

Now by a calculation as in the proof of Theorem 4.1 we have

$$X_t^{1/n} = X_0 \int_0^t \bar{H}_s ds + \frac{1}{n^\beta} U_t\,, \tag{5.21}$$

$$Y_t^{1/n} = \int_0^t \bar{h}_s ds + \frac{1}{n^\alpha} W_t\,, \tag{5.22}$$

where

$$\bar{H}_s = H_s + \beta\psi_t^{1/n}, \quad \psi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\beta dU_s\,,$$

$$\bar{h}_s = h_s + \alpha\varphi_t^{1/n}, \quad \varphi_t^{1/n} = \int_0^t (t - s + \frac{1}{n})^\alpha dW_s\,.$$

By the Ito isometry we can see that:

$$\int_0^t E\bar{H}_s^2 ds < \infty \text{ and } \int_0^t E\bar{h}_s^2 ds < \infty\,. \tag{5.23}$$

Then we can write the FKK filtering equation for the approximate model (5.21)-(5.22)-(5.23) as in (5.16), where $\nu_t^{1/n}$ is the innovation process

$$\nu_t^{1/n} = Y_t^{1/n} - \int_0^t \pi_s^{(n)}(\bar{h})ds.$$

Here $\pi_t^{(n)}(f) = E(f(X_t^{1/n})|\mathcal{F}_t^{Y_t^{1/n}})$.

Because $X_t^{1/n} \to X_t$ and $Y_t^{1/n} \to Y_t$ in L^2 and also

$$\|E\big(f(X_t^{1/n})|\mathcal{F}_t^{B^{(2)1/n}}\big) - E\big(f(X_t^{1/n})|\mathcal{F}_t^{Z^{1/n}}\big)\|_{L^2} \to 0 \text{ as } n \to \infty\,, \tag{5.24}$$

where $Z^{1/n} = \int\limits_0^t h_s ds + B^{(2)}_{t+\frac{1}{n}}$, so an analogous assertion to the proof of Theorem 3.1 says that $\pi_t(f) = L^2\text{-}\lim\limits_{n\to\infty} \pi_t^{(n)}(f)$ exists where $\pi_t^{(n)}$ satisfies the FKK filtering equation (5.16). $\square$

References

[1] Coutin, L. and Decreusefond, L., *Abstract non-linear filtering theory in presence of fractional Brownian motion*, Ann. App. Probab. Vol 9, No. **4**(1999), 1058-1090.

[2] Dai, W. and Heyde, C.C., *Itô's formula with respect to fractional motion and its applications,* Journal of Applied Mathematics and Stochastic Analysis, 9, No **4**(1996), 439-448.

[3] Decreusefond, L. and Üstunel, A.S., *Stochastic analysis of the fractional Brownian motion*, Potential Analysis, **10**(1999), 177-214.

[4] Ciesielski, Z., Kerkyacharian, G. and Roynette, B., *Quelques espaces fonctionels associés à des processus gaussiens*, Studia Mathematica, **107(2)**(1993), 171-204.

[5] Loève, M., *Probability theory* (4th edn.), Springer-Velag, 1977.

[6] Mandelbrot, B. and Van Nees, H., *Fractional Brownian motion, fractional noises and applications*, SIAM Rev, Vol. 10(1968), 422-436.

[7] Revuz, D. and Yor, M., *Continuous martingales and Bronian motion*, Springer, 3rd Edition, 1998.

[8] Thao, T.H., *An approximate approach to fractional analysis for finance*, Nonlinear Analysis, Real World Applications, Vol 7, No. **1**(2006), 127-13, 2006. *Available online on ScienceDirect.*

Institute of Mathematics,
Vien Toan-Hoc, Vast,
Vietnam

School Of Mathematics
Suranaree University of Technology,
Thailand

School Of Mathematics
Suranaree University of Technology,
Thailand