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Fractional integrated GARCH diffusion limit models[☆]

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ABSTRACT

In this paper, we introduce an approximate approach to the fractional integrated GARCH(1,1) model of continuous time perturbed by fractional noise. Based on the L^2-approximation of this noise by semimartingales, we proved a convergence theorem concerning an approximate solution. A simulation example shows a significant reduction of error in a fractional stock price model as compared to the classical stock price model.

1. Introduction

Risk in a financial market is measured by using volatility. So predictability of volatility has important implications for risk management. If volatility increases, so will Value At Risk (VAR). Investors may want to adjust their portfolio to reduce their exposure to those assets whose volatility is predicted to increase. One method that is widely employed for volatility estimation is to use GARCH models. A discrete time GARCH(1,1) model is a model of the form

$$\nu_{k+1} = \omega_0 + \beta\nu_{k+1} + \alpha\nu_k U_k^2, \qquad X_k = \sigma_k U_k \tag{1}$$

where $\sigma_k = \sqrt{\nu_k}$, and α, β are weight parameters, ω_0 is a parameter related to the long-term variance, and U_K is a sequence of independent normal random variables with zero mean and variance of 1.

It is well known that GARCH models are not designed for long range-dependence (LRD). But there are some empirical studies showing that log-return series (X_t) of foreign exchange rates, stock indices and share prices exhibit the LRD effect (see, for example, Mikosch and Starica (2003, page 445)). In 1990, Nelson (1990) showed that as the time interval decreases and become infinitesimal, Eq. (1) can be changed to

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}W_t \tag{2}$$

where $v_t = \sigma_t^2$ is the stock-return variance, ω, θ and ξ are weight parameters and W_t is a standard Brownian motion process. To be more accurate, there is a weak convergence of the discrete GARCH process to the continuous diffusion limit. The purpose of this paper is to introduce LRD effect into GARCH models of continuous time (i.e., into Eq. (2)). The importance of this process in finance is that it can be used to forecast volatility and risk of some financial instruments.

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Recall that a fractional Brownian motion process W_t^H, with Hurst index H, is a centered Gaussian process such that its covariance function $R(t,s) = EW_t^H W_s^H$ is given by

$$R(s,t) = \frac{1}{2}(|t|^\gamma + |s|^\gamma - |t-s|^\gamma)$$

where $\gamma = 2H$ and $0 < H < 1$. If $H = \frac{1}{2}$, then W_t^H is the usual standard Brownian motion process. For $H \neq \frac{1}{2}$, W_t^H is neither a semimartingale nor a Markov process so we cannot apply the standard stochastic calculus for this process. It is a process of long range dependence in the following sense: If $\rho_n = E[W_1^H(W_{n+1}^H) - W_n^H]$, then the series $\sum_{n=0}^\infty \rho_n$ is either divergent or convergent with very late rate. It is known that a fractional Brownian motion W_t^H can be decomposed as follows:

$$W_t^H = \frac{1}{\Gamma(1+\alpha)}\left[Z_t + \int_0^t (t-s)^\alpha \mathrm{d}W_s\right],$$

where Γ is the gamma function, $Z_t = \int_{-\infty}^0 [(t-s)^\alpha - (-s)^\alpha]\mathrm{d}W_s$, $\alpha = H - \frac{1}{2}$, and W_t is a standard Brownian motion.

We suppose from now on $0 < \alpha < \frac{1}{2}$ so that $\frac{1}{2} < H < 1$. Then Z_t has absolutely continuous trajectories and it is the term $B_t^H := \int_0^t (t-s)^\alpha \mathrm{d}W_s$ that exhibits long range dependence. We will use B_t^H instead of W_t^H in fractional stochastic calculus.

In Thao (2006) constructed an approximate process B_t^ε of B_t^H as follows:

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} \mathrm{d}W_s$$

where $\frac{1}{2} < H < 1$, and W_t is a standard Brownian motion. He also proved that $B_t^\varepsilon \to B_t^H$ in $L^2(\Omega)$ as $\varepsilon \to 0$ (uniformly in t) and B_t^ε is a semimartingale. These results give us a convenient way to study fractional Brownian motions since we can use the standard Ito integrals and then it is easy to do numerical approximation.

By a fractional integrated GARCH model of continuous time (FIGARCH), we shall mean a process of the form

$$\mathrm{d}v_t = (\omega - \theta v_t)\mathrm{d}t + \xi v_t \mathrm{d}B_t^H \tag{3}$$

where $0 \le t \le T$, ω, θ and ξ are weight parameters, and B_t^H is a fractional Brownian motion. For each $\epsilon > 0$, an approximate model of the FIGARCH model is a process of the form

$$\mathrm{d}v_t^\varepsilon = (\omega - \theta v_t^\varepsilon)\mathrm{d}t + \xi v_t^\varepsilon \mathrm{d}B_t^\varepsilon \tag{4}$$

where B_t^ε is the approximate process of B_t^H. We shall show that its solution converges to the solution of the FIGARCH model (3).

Moreover, geometric Brownian motion for the asset price was used to simulate the SCB stock prices where the volatility of this model was predicted from an approximate fractional variance process of GARCH(1,1) model in continuous time and classical GARCH(1,1) model in continuous time. And both of them were compared with the empirical historical stock prices of SCB.

2. Solutions of the approximate models

In this section, we are interested in finding a solution of the approximate model (4) together with initial condition $v_{t(t=0)}^\varepsilon = v_0$.

Let $\varepsilon > 0$. Recall that an approximated process B_t^ε is defined by

$$B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \mathrm{d}W_s,$$

where $\alpha = H - \frac{1}{2}$, $0 < H < 1$, and W_t is a Brownian motion process. By an application of the stochastic Fubini Theorem, one gets

$$\begin{aligned}\int_0^t \int_0^s (s-u+\varepsilon)^{\alpha-1}\mathrm{d}W_u \mathrm{d}s &= \int_0^t \int_u^t (s-u+\varepsilon)^{\alpha-1}\mathrm{d}s\mathrm{d}W_u \\ &= \frac{1}{\alpha}\int_0^t ((t-u+\varepsilon)^\alpha - \varepsilon^\alpha)\mathrm{d}W_u \\ &= \frac{1}{\alpha}\left[\int_0^t (t-u+\varepsilon)^\alpha \mathrm{d}W_u - \varepsilon^\alpha \int_0^t \mathrm{d}W_u\right] \\ &= \frac{1}{\alpha}[B_t^\varepsilon - \varepsilon^\alpha W_t].\end{aligned}$$

Consequently

$$B_t^\varepsilon = \alpha \int_0^t \varphi_s^\varepsilon \mathrm{d}s + \varepsilon^\alpha W_t$$

where

$$\varphi_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{\alpha-1} \mathrm{d}W_s.$$

Thus we have

$$\mathrm{d}B_t^\varepsilon = \alpha\varphi_t^\varepsilon \mathrm{d}t + \varepsilon^\alpha \mathrm{d}W_t. \tag{5}$$

Substituting $\mathrm{d}B_t^\varepsilon$ from (5) into (4), then Eq. (4) can be rewritten into the following form

$$\begin{aligned} \mathrm{d}v_t^\varepsilon &= (\omega - \theta v_t^\varepsilon)\mathrm{d}t + \xi v_t^\varepsilon(\alpha\varphi_t^\varepsilon \mathrm{d}t + \varepsilon^\alpha \mathrm{d}W_t), \\ &= (\omega - \theta v_t^\varepsilon + \xi\alpha\varphi_t^\varepsilon v_t^\varepsilon)\mathrm{d}t + \xi v_t^\varepsilon \varepsilon^\alpha \mathrm{d}W_t. \end{aligned} \tag{6}$$

Theorem 1. *For any $\varepsilon > 0$, a solution of the approximate model (4) is given by*

$$v_t^\varepsilon = \exp\left(\xi B_t^\varepsilon - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t\right)\left(v_0^\varepsilon + \omega\int_0^t \mathrm{e}^{\left(\theta+\frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)s - \xi B_s^\varepsilon}\mathrm{d}s\right), \tag{7}$$

where $-\frac{1}{2} < \alpha < \frac{1}{2}$ and $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^\alpha \mathrm{d}W_s$.

Proof. To find a solution of (6), we look for a solution of the form

$$v_t^\varepsilon = U_t V_t$$

where

$$\mathrm{d}U_t = (-\theta + \xi\alpha\varphi_t^\varepsilon)U_t \mathrm{d}t + \xi\varepsilon^\alpha U_t \mathrm{d}W_t$$

and

$$\mathrm{d}V_t = a_t \mathrm{d}t + b_t \mathrm{d}W_t.$$

Firstly, we shall find a solution of $\mathrm{d}U_t = (-\theta + \xi\alpha\varphi_t^\varepsilon)U_t \mathrm{d}t + \xi\varepsilon^\alpha U_t \mathrm{d}W_t$. By an application of the Ito formula to the function $f(u) = \ln u$ for $u = U_t$, one gets

$$\begin{aligned} \mathrm{d}\,(\ln U_t) &= \frac{1}{U_t}\mathrm{d}U_t - \frac{1}{2U_t^2}(\mathrm{d}U)^2 \\ &= \frac{1}{U_t}\left((-\theta + \xi\alpha\varphi_t^\varepsilon)U_t \mathrm{d}t + \xi\varepsilon^\alpha U_t \mathrm{d}W_t\right) - \frac{1}{2U_t^2}(\xi^2\varepsilon^{2\alpha}U_t^2 \mathrm{d}t) \\ &= \left(-\theta + \xi\alpha\varphi_t^\varepsilon - \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)\mathrm{d}t + \xi\varepsilon^\alpha \mathrm{d}W_t \end{aligned}$$

or, equivalently,

$$\ln U_t - \ln U_0 = \xi\alpha\int_0^t \varphi_s^\varepsilon \mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t + \xi\varepsilon^\alpha W_t.$$

That is

$$U_t = U_0 \exp\left(\xi\alpha\int_0^t \varphi_s^\varepsilon \mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t + \xi\varepsilon^\alpha W_t\right). \tag{8}$$

Set $U_0 = 1$ and $V_0 = v_0^\varepsilon$. Taking the differential of the product, we get

$$\begin{aligned} \mathrm{d}\,(U_t V_t) &= U_t \mathrm{d}V_t + V_t \mathrm{d}U_t + \mathrm{d}U_t \mathrm{d}V_t \\ &= U_t\,(a_t \mathrm{d}t + b_t \mathrm{d}W_t) + V_t\left((-\theta + \xi\alpha\varphi_t^\varepsilon)U_t \mathrm{d}t + \xi\varepsilon^\alpha U_t \mathrm{d}W_t\right) + \xi\varepsilon^\alpha U_t b_t \mathrm{d}t \\ &= \left(U_t a_t + V_t(-\theta + \xi\alpha\varphi_t^\varepsilon)U_t + \xi\varepsilon^\alpha U_t b_t\right)\mathrm{d}t + (U_t b_t + V_t\xi\varepsilon^\alpha U_t)\,\mathrm{d}W_t. \end{aligned}$$

Since $v_t^\varepsilon = U_t V_t$ then

$$\mathrm{d}v_t^\varepsilon = \left(U_t a_t + (-\theta + \xi\alpha\varphi_t^\varepsilon)v_t^\varepsilon + \xi\varepsilon^\alpha U_t b_t\right)\mathrm{d}t + \left(U_t b_t + \xi\varepsilon^\alpha v_t^\varepsilon\right)\mathrm{d}W_t. \tag{9}$$

Comparing the coefficients of Eq. (9) with Eq. (6), we see that the desired coefficients a_t and b_t turn out to satisfy the following equations

$$U_t b_t = 0$$

and

$$U_t a_t + \xi \varepsilon^\alpha U_t b_t = \omega.$$

Then $b_t = 0$ and $a_t = \frac{\omega}{U_t}$. Hence

$$V_t := V_0 + \int_0^t a_t \mathrm{d}t + \int_0^t b_t \mathrm{d}W_t = v_0^\varepsilon + \int_0^t \frac{\omega}{U_s}\mathrm{d}s.$$

Moreover, v_t^ε is found to be

$$v_t^\varepsilon := U_t V_t = U_t\left(v_0^\varepsilon + \int_0^t \frac{\omega}{U_s}\mathrm{d}s\right).$$

Hence, with $U_0 = 1$ and using U_t as in (8), the solution of v_t^ε is given by

$$v_t^\varepsilon = \exp\left(\xi\alpha\int_0^t \varphi_s^\varepsilon \mathrm{d}s - \left(\theta + \frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)t + \xi\varepsilon^\alpha W_t\right)\left(v_0^\varepsilon + \omega\int_0^t \mathrm{e}^{\left(\theta+\frac{1}{2}\xi^2\varepsilon^{2\alpha}\right)s - \xi\varepsilon^\alpha W_s - \xi\alpha\int_0^s \varphi_u^\varepsilon \mathrm{d}u}\mathrm{d}s\right).$$

This proves Theorem 1. ■

3. Convergence of the solutions of an approximate model

To prove the convergence of v_t^ε, firstly, let us consider the process v_t which satisfies Eq. (2). Let $X_t = \ln v_t$. It follows from the Ito formula that

$$\mathrm{d}X_t = \left(\omega \mathrm{e}^{-X_t} - \frac{\xi^2}{2} - \theta\right)\mathrm{d}t + \xi \mathrm{d}W_t. \tag{10}$$

The fractional model of the process X_t is a process which is of the form

$$\mathrm{d}X_t = \left(\omega \mathrm{e}^{-X_t} - \frac{\xi^2}{2} - \theta\right)\mathrm{d}t + \xi \mathrm{d}B_t^H, \tag{11}$$

where B_t^H is a fractional Brownian motion. Then an approximated model of (11) is of the form

$$\mathrm{d}X_t^\varepsilon = \left(\omega \mathrm{e}^{-X_t^\varepsilon} - \frac{\xi^2}{2} - \theta\right)\mathrm{d}t + \xi \mathrm{d}B_t^\varepsilon \tag{12}$$

where B_t^ε has already been defined in Section 1.

Theorem 2. *The solution of (12) converges to the solution of (11) in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$.*

Proof. We note that Eqs. (11) and (12) give

$$X_t - X_t^\varepsilon = \omega\int_0^t \left(\mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^\varepsilon}\right)\mathrm{d}s + \xi\left(B_t^H - B_t^\varepsilon\right). \tag{13}$$

Let $\|\cdot\|$ denote the norm in $L^2(\Omega)$. It follows from (13) that

$$\begin{aligned}\left\|X_t - X_t^\varepsilon\right\| &= \left\|\omega\int_0^t \left(\mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^\varepsilon}\right)\mathrm{d}s + \xi\left(B_t^H - B_t^\varepsilon\right)\right\| \\ &\leqslant |\omega|\int_0^t \left\|\mathrm{e}^{-X_s} - \mathrm{e}^{-X_s^\varepsilon}\right\|\mathrm{d}s + |\xi|\left\|B_t^H - B_t^\varepsilon\right\|.\end{aligned}$$

Since e^{-x} is differentiable and bounded on every compact interval, then

$$\left\|X_t - X_t^\varepsilon\right\| \leqslant |\omega|\int_0^t K_1\left\|X_s - X_s^\varepsilon\right\|\mathrm{d}s + |\xi|\left\|B_t^H - B_t^\varepsilon\right\| \tag{14}$$

for some constants $K_1 > 0$. Referring to a result Thao (2006, page 127), one gets

$$\left\|B_t^H - B_t^\varepsilon\right\| \le C(\alpha)\varepsilon^{\frac{1}{2}+\alpha}, \tag{15}$$

where $0 < \alpha < \frac{1}{2}$ for $\frac{1}{2} < H < 1$ and $-\frac{1}{2} < \alpha < 0$ for $0 < H < \frac{1}{2}$, and $C(\alpha)$ is a positive constant depending only on α.

It follows from (14) and (15) that

$$\left\|X_t - X_t^{\varepsilon}\right\| \leqslant |\omega| K_1 \int_0^t \left\|X_s - X_s^{\varepsilon}\right\| \mathrm{d}s + |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha}. \tag{16}$$

Applying Gronwall's lemma to (16), then

$$\left\|X_t - X_t^{\varepsilon}\right\| \leqslant |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 t}.$$

Therefore

$$\sup_{0 \le t \le T} \left\|X_t - X_t^{\varepsilon}\right\| \leqslant |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 T} \to 0$$

as $\varepsilon \to 0$. So $X_t^{\varepsilon} \to X_t$ in $L^2(\Omega)$ as $\varepsilon \to 0$ and uniformly with respect to t. ■

Theorem 3. *If $X_t^{\varepsilon} \to X_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$, then $v_t^{\varepsilon} \to v_t$ in $L^2(\Omega)$ uniformly with respect to $t \in [0, T]$ as $\varepsilon \to 0$.*

Proof. It follows from $X_t = \ln v_t$, so $v_t = \mathrm{e}^{X_t}$ that

$$\left\|v_t - v_t^{\varepsilon}\right\| = \left\|\mathrm{e}^{X_t} - \mathrm{e}^{X_t^{\varepsilon}}\right\|.$$

Since e^x is differentiable and bounded in some closed interval, then

$$\left\|v_t - v_t^{\varepsilon}\right\| \le K_2 \left\|X_t - X_t^{\varepsilon}\right\|$$

for some positive constant K_2. From (15), we obtain

$$\left\|v_t - v_t^{\varepsilon}\right\| \leqslant K_2 |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 t}.$$

Therefore

$$\sup_{0 \le t \le T} \left\|v_t - v_t^{\varepsilon}\right\| \leqslant K_2 |\xi| C(\alpha) \varepsilon^{\frac{1}{2}+\alpha} \mathrm{e}^{|\omega| K_1 T} \to 0$$

as $\varepsilon \to 0$. The proof is now complete. ■

4. Applications

In this section, volatilities of the stock of Siam Commerical Bank (SCB) are computed by using FIGARCH(1,1) model and classical GARCH(1,1) model of continuous time. Then SCB stock prices are simulated by using these volatilities. After that both simulated SCB stock prices are compared with the empirical historical prices of SCB.

4.1. SCB simulated stock prices

A model for the dynamic of an asset price that will be considered here is of the form

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t,$$

where μ is known as the drift rate or rate of return of the price S_t and W_t is a Brownian motion. The stochastic volatility σ_t (which measures the standard deviation of the return $\frac{\mathrm{d}S_t}{S_t}$) is defined by $\sigma_t := \sqrt{v_t}$ where v_t is the FIGARCH model of continuous time as in Eq. (3). For comparative purposes, we shall compute the percentage error (PE) between two sets of data by the following formula

$$PE = \frac{1}{K} \sum_{i=1}^{K} \frac{|X_i - Y_i|}{X_i} \times 100,$$

where K is sample size, $X = (X_i, i \ge 1)$ is the market prices and $Y = (Y_i, i \ge 1)$ is the model prices. We use $K = 245$ when we sample data for 12 months.

For simulation purposes, we consider an approximate model

$$\mathrm{d}S_t^{\varepsilon} = \mu S_t^{\varepsilon} \mathrm{d}t + \sigma_t^{\varepsilon} S_t^{\varepsilon} \mathrm{d}W_t, \tag{17}$$

with $\epsilon > 0$ and $\sigma_t^{\varepsilon} = \sqrt{v_t^{\varepsilon}}$. The fractional variance process v_t^{ε} will be simulated by using Eq. (7), i.e.,

$$v_t^{\varepsilon} = \exp\left(\xi B_t^{\varepsilon} - \left(\theta + \frac{1}{2}\xi^2 \varepsilon^{2\alpha}\right) t\right) \left(v_0^{\varepsilon} + \omega \int_0^t \mathrm{e}^{\left(\theta + \frac{1}{2}\xi^2 \varepsilon^{2\alpha}\right)s - \xi B_s^{\varepsilon}} \mathrm{d}s\right). \tag{18}$$

Table 1
Discrete parameters ω_h, β_h and α_h of each dataset.

Months	Dataset (DD/MM/YY)	ω_h	β_h	α_h
1	1/12/2006–28/12/2006	0.0033	0	0.1689
3	2/10/2006–28/12/2006	0.0012	0	0.1154
6	3/7/2006–28/12/2006	0.00077659	0	0.0887
9	3/4/2006–28/12/2006	0.00071251	0	0.0726
12	3/1/2006–28/12/2006	0.00062672	0	0.0692

Table 2
Parameters ω, θ and ξ obtained from each dataset.

Months	Dataset (DD/MM/YY)	ω	θ	ξ
1	1/12/2006–28/12/2006	0.0033	0.8311	0.2389
3	2/10/2006–28/12/2006	0.0012	0.8846	0.1632
6	3/7/2006–28/12/2006	0.00077659	0.9113	0.1254
9	3/4/2006–28/12/2006	0.00071251	0.9274	0.1027
12	3/1/2006–28/12/2006	0.00062672	0.9308	0.0979

Table 3
Average *PE* for each set of parameters.

Parameters	ω	θ	ξ	Average of *PE* (%)
1	0.0033	0.8311	0.2389	40.5099
2	0.0012	0.8846	0.1632	24.0161
3	0.00077659	0.9113	0.1254	19.3196
4	0.00071251	0.9274	0.1027	18.3600
5	**0.00062672**	**0.9308**	**0.0979**	**17.3366**

The actual stock prices of Siam Commercial Bank (SCB) were obtained from http://www.tiscoetrade.com. Using the dataset from January 3, 2006 to December 28, 2007. We divide these data into two disjoint sets. The first one, from January 3, 2006 to December 28, 2006, will be used to estimate parameters ω, θ, and ξ for Eq. (18). The second set (January 3, 2007–December 28, 2007) will be used for comparison with the simulated prices.

We begin by estimating parameters ω, θ and ξ. To do this, we firstly enter the following 5 datasets, i.e., 1 month (December 1, 2006–December 28, 2006), 3 months (October 2, 2006–December 28, 2006), 6 months (July 3, 2006–December 28, 2006) and 12 months (January 3, 2006–December 28, 2006) into Matlab 6.5 (GARCH Toolbox) with COMPAQ Presario B1908TU to obtain discrete parameters of GARCH(1,1) model (ω_h, β_h and α_h). Those discrete parameters from each datasets are shown in Table 1.

Secondly, we utilize the formulas between discrete parameters and continuous parameters which have been given by Nelson (1990) to estimate the parameters ω, θ and ξ. The formulas are as follows:

$$\omega = h^{-1}\omega_h,$$
$$\theta = h^{-1}(1 - \beta_h - \alpha_h),$$
$$\xi = \sqrt{2h^{-1}}\alpha_h,$$

where h is the time lag between two consecutive data. Here we use $h = 1$. Thus the estimated parameters ω, θ and ξ for each dataset (1, 3, 6, 9 and 12 months) are given in Table 2.

From the information in Table 2, we look for those parameters which can give us the mimum average of *PE*. In order to solve this problem, we simulated v_t^ε (see, Eq. (18)) by using the parameters ω, θ and ξ from each dataset (1, 3, 6, 9 and 12 months). Here, we choose $\varepsilon = 0.0001$, $\alpha = 0.15$, $\mu = 0.0017819$ and $v_0^\varepsilon = 0$. Then, by using $\sigma_t^\varepsilon = \sqrt{v_t^\varepsilon}$, the SCB stock prices from January 3, 2007 to December 28, 2007 were forecast by the pricing model S_t^ϵ (see, Eq. (17)). Next, we compute *PE* by using the information from the simulation and the empirical data of SCB closing prices (January 3, 2007–December 28, 2007). For each set of parameters, we calculated the average of *PE* for 5000 paths. The results are shown in Table 3.

It can be seen from Table 3 that the parameters $\omega = 0.00062672$, $\theta = 0.9308$ and $\xi = 0.0979$ give us the minimum value of the average *PE*. We select this set of parameters for forecasting the future stock prices of SCB. In summary, when the SCB stock prices were simulated by Eq. (17) using parameters as mentioned above, the average of *PE* and its variance will be given as follows:

$$\begin{aligned} &\text{average of } PE = 17.3366\% \\ &\text{variance} = 43.0287\%. \end{aligned} \tag{19}$$

Recall that a GARCH(1,1) model of continuous time is of the form

$$dv_t = (\omega - \theta v_t)dt + \xi v_t dW_t, \tag{20}$$

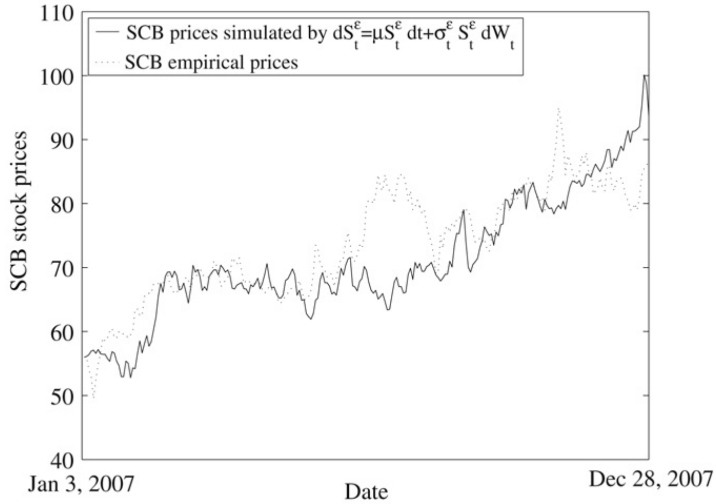

Fig. 1. Price behaviour of SCB, between January 3, 2007 and December 28, 2007, compared with a scenario simulated from fractional pricing model (dashed line := empirical data, solid line := simulated by $\mathrm{d}S^\varepsilon_t = \mu S^\varepsilon_t \mathrm{d}t + \sigma^\varepsilon_t S^\varepsilon_t \mathrm{d}W_t$, $PE = 6.0401\%$).

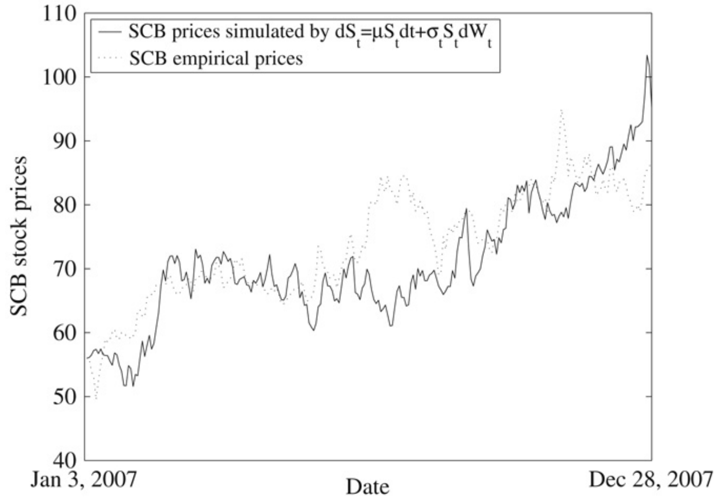

Fig. 2. Price behaviour of SCB, between January 3, 2007 and December 28, 2007, compared with a scenario simulated by pricing model (dashed line := empirical data, solid line := simulated by $\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t$, $PE = 6.9627\%$).

and the pricing model is

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \tag{21}$$

where $\sigma_t = \sqrt{v_t}$.

We simulated the pricing model (21) by using $\omega = 0.00062672$, $\theta = 0.9308$, $\xi = 0.0979$, $\mu = 0.0017819$, $v_0 = 0$ and $K = 245$. We compute the *PE* of these simulation prices and the empirical data of SCB closing prices from January 3, 2007 to December 28, 2007. Next we compute the average of *PE*, by using $N = 5000$, and found that

$$\begin{aligned} &\text{average of } PE = 21.6536\% \\ &\text{variance} = 69.2135\%. \end{aligned} \tag{22}$$

By comparing the average *PE* and its variance by Eq. (19) and (22), one can see that in the case of SCB, the forecast of the future stock prices by using model (17) (which includes the fractional part) give an average error significantly smaller than using model (21) (which does not includes the fractional part).

For an illustration, Fig. 1 shows the empirical data of SCB as compared to the price simulated by the fractional price model (17). Here we used $\varepsilon = 0.0001$, $\alpha = 0.15$, $\theta = 0.9308$, $\omega = 0.00062672$, $\xi = 0.0979$ and $v^\varepsilon_0 = 0$. The percentage error $PE = 6.0401\%$.

Fig. 2, shows the empirical data of SCB as compared to the price simulated by the price model (21). Here we used $\theta = 0.9308$, $\omega = 0.00062672$, $\xi = 0.0979$, $\mu = 0.0017819$, $v_0 = 0$ and $\sigma_t = \sqrt{v_t}$. The percentage error $PE = 6.9627\%$.

By comparing *PE* and variance of Figs. 1 and 2, one can see that in the case of SCB the sample path from the fractional pricing model gives a better fit with the data than the ordinary pricing model, since the percentage error is smaller.

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