ชั้นของสมการวิวัฒนาการกึ่งเชิงเส้น และการควบคุมเหมาะที่สุด

นาย อนุสรณ์ ชนวีระยุทธ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
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A CLASS OF SEMILINEAR EVOLUTION EQUATIONS AND OPTIMAL CONTROL

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ในวิทยานิพนธ์ฉบับนี้ ได้ศึกษา การมีจริงเฉพาะที่ ความเป็นไปได้อย่างเดียว ภาคขยาย การมีจริงวงกว้างของผลเฉลยไมลด์ สำหรับชั้นของสมการวิวัฒนาการกึ่งเชิงเส้นที่มีการประวิงในปริภูมิบานาค ยิ่งกว่านั้น ได้พิจารณาปัญหาการควบคุมเหมาะที่สุดแบบโบลซาของระบบควบคุมที่สมนัยกัน

ได้พิสูจน์การมีจริงเฉพาะที่ของผลเฉลยไมลด์ ได้พิสูจน์ทฤษฎีบทภาคขยายโดยใช้ค่าประมาณก่อนประสบการณ์ ได้พิสูจน์บทตั้งของกรอนวัลที่มีภาวะเอกฐานและการล้าหลังเชิงเวลา เพื่อเป็นเครื่องมือสำหรับการได้ค่าประมาณก่อนประสบการณ์ ได้เพิ่มเงื่อนไขการเติบโตเชิงเส้นเพื่อพิสูจน์ทฤษฎีบทการมีจริงวงกว้าง ยิ่งกว่านั้นได้พิสูจน์ทฤษฎีบทการมีจริงวงกว้างที่ทั่วไปภายใต้เงื่อนไขการเติบโตเชิงเส้นซูเปอร์ ได้ศึกษาความไม่อิสระอย่างต่อเนื่องของระบบและการมีจริงของผลเฉลยไมลด์สำหรับระบบที่มีการประวิงอนันต์ ได้พิจารณาความปรกติของผลเฉลยไมลด์

ได้ศึกษาการมีจริงของผลเฉลยไมลด์สำหรับระบบควบคุมที่ทั่วไปกว่า และนำเสนอการมีจริงของความเหมาะที่สุดสำหรับปัญหาการควบคุมเหมาะที่สุดแบบโบลซา โดยใช้ผลลัพธ์ของบัลเดอร์

ท้ายสุด ได้อธิบายผลลัพธ์เชิงนามธรรมให้เข้าใจ ด้วยสองตัวอย่างซึ่งเกี่ยวข้องกับสมการเชิงอนุพันธ์ย่อยแบบพาราโบลากึ่งเชิงเส้นที่มีการประวิงจำกัดและปัญหาการควบคุมเหมาะที่สุดที่สมนัยกัน

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ANALYTIC SEMIGROUP, MILD SOLUTIONS, A PRIORI ESTIMATE, GRONWALL'S LEMMA, OPTIMAL CONTROL

In this thesis, local existence, uniqueness, extension, and global existence of mild solutions for a classical of semilinear evolution equations with delay in Banach spaces are investigated. Moreover, Bolza optimal control problem of a corresponding controlled system is also considered.

Local existence of mild solutions is proved. Extension theorem is also proved by a priori estimate. Gronwall's lemma with singularity and time lag is derived to be a tool for obtaining a priori estimate. Linear growth condition is implemented to prove global existence theorem. Moreover, a general global existence theorem is proved under super linear growth condition. Continuous dependence of the system and existence of mild solutions for a system with infinite delay are investigated. Regularity of mild solutions is considered.

Existence of mild solutions for a more general controlled system is investigated. Existence of optimality for Bolza optimal control problem is presented by using a Balder's result.

Finally, the abstract results are illustrated by two examples concerning semilinear parabolic partial differential equations with finite delay and corresponding optimal control problem.

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Anusorn Chonweerayuth

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Chapter I

Introduction

Many phenomena in the real world can be described by infinite dimensional systems, for instance; heat conduction, properties of elastic material, fluid dynamics, diffusion-reaction processes, etc.. The variable that we are studying (temperature, displacement, concentration, velocity, etc.) is usually referred to as the *state*. The space in which the state exists is called the *state space*, and the equation that the state satisfies is called the *state equation* which may be one of the following types: partial differential equation, functional differential equation, integrodifferential equation, or abstract evolution equation. Stochastic differential equation is also an infinite dimensional system.

It is well known that several classes of differential equations with memory effects can be formulated as abstract semilinear evolution equation with a delay or retardation, i.e., the equation evolved with time and the principal part of their differential operators are linear and other terms are nonlinear with respect to a variable in a suitable function space and the unknown function depends on a delay or historical effects. We sometimes call those evolution equations as a system and want to study many properties of their solutions.

Most of the system concern with many types of solutions, for instance, classical solution, weak solution, strong solution, mild solution, and others. So the meaning of solution should be defined and the existence of the solution is a fundamental problem that we should answer before we study other properties of the solution, e. g., uniqueness, continuous dependence on initial data, stability, etc.

In the seventeenth century, Bernoulli studied the brachistochrone problem, and subsequently initiated the classical calculus of variations. After three hundred years of evolution, optimal control theory has been formulated as a generalized extension of the calculus of variations.

A system can be controlled by supplying some control function or control policy to achieve some purpose. We call the system the controlled system. Optimal control problem is to find a control policy to minimize or maximize some objective functional subject to a dynamic framework.

In this thesis, we consider semilinear integro-differential equations with time lags on a Banach space X. The systems are

$$\begin{cases} \dfrac{d}{dt}x(t) + Ax(t) = f(x(t)) + \int_{-r}^{t} h(t-s)g(x(s))ds, t \in [0,T], \\ x(t) = \varphi(t), t \in [-r,0], \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{d}{dt}x(t) + Ax(t) = f(t, x(t)) + \int_{-r}^{t} h(t-s)g(s, x(s))ds, t \in [0,T], \\ x(t) = \varphi(t), t \in [-r,0]. \end{cases} \tag{1.2}$$

We systematically study local existence, extension, global existence and regularity of mild solutions. Continuous dependence on initial conditions of those mild solutions and existence theorem for infinite delay system are investigated. The semigroup theory, especially analytic semigroup and fractional powers of operator, and the contraction mapping theorem (or the Banach fixed point theorem) are used to obtain our results (See Ahmed N. U. (1991), Pazy A. (1983)). Existence of an optimal control and Bolza optimal control problem are studied. Some examples are presented to complete our work.

Many authors studied semilinear evolution equations (See Li, X. and Yong, J. (1995), Ahmed N. U. (1991), Amann, H. (1978)). Some study semilinear evolution equations with delay (See Wu, J. (1996), Xiang, X., Kuang, H. (2000)). Ahmed, N. U. (1991) gives a result about global existence and uniqueness of mild solutions for an integrodifferential equation (1.1). In his results, uniform Lipschitz condition is too strong for discussion of global existence. We will show that by using a weaker condition, locally Lipchitz condition is enough to guarantee local existence of mild solutions, and by adding some growth conditions, global existence problem can be solved. In Amann, H. (1978), he also study local and global existence of mild solutions for semilinear evolution equations without delay effects. He use an infinitesimal generator A(t) depending on t. We extend some results in his works to delay systems.

We consider existence problems in several kinds of situations (See assumption (A), (F1)-(F6), (G1)-(G6), (H1)-(H2) in Chapter III) that are different from others. It is well known that a priori estimate is a very important condition to prove extension theorem. A difficulty has been occurred for giving a priori estimate, because Gronwall' s inequality is without delay term, so it cannot be directly used to derive the a priori estimate in our cases. So we derived a Gronwall's lemma with singularity and time lag that is suitable for our system. We use the Gronwall's lemma and nearly linear growth condition to obtain a priori estimate. In addition, we use the Moment inequality under super linear growth condition to obtain a priori estimate for global existence problem.

Regularity of mild solutions is also discussed by using technique of fractional power operators. Continuous dependence of our system is investigated. Our method is easy to extend to semilinear evolution equations with infinite delay.

Moreover, we use abstract results about existence of mild solutions to study the existence of an optimal control for the controlled system corresponding to system (1.1). We consider the Bolza controlled problem, that is to minimize the functional J, on the admissible control set U_{ad}, defined by

$$J(u) = \int_I \ell(t, x^u(t), u(t))\,dt + \psi(x(T)),$$

where ℓ is a function satisfying some properties, ψ is a nonnegative function. We show how Balder's theorem can be applied.

We give some examples that illustrate our abstract results. These examples show how to apply our main results to semilinear parabolic controlled systems.

The thesis is organized as follows: Chapter II mainly introduces theoretical backgrounds and provides the convenient references to the well known facts of differential equations on Banach space. Chapter III deals with local existence and uniqueness of mild solutions, extension theorem, global existence theorem, regularity of mild solutions, continuous dependence on initial conditions, existence of mild solutions of a system with infinite delay. Chapter IV deals with existence of an optimal control of Bolza problem . In chapter V, some examples are presented to demonstrate the applicability of our abstract results. We conclude all results found in chapter VI.

Chapter II

Preliminaries

In this chapter, we present some important definitions and theorems which are useful for understanding the results that appear in the following chapters.

2.1 Semigroups

For Banach spaces X and Y, let L(X, Y) denote the class of all linear and bounded operators from X into Y, and L(X) for L(X, X).

Definition 2.1.1. Let X be a Banach space. A one parameter family $\{T(t) \mid 0 \le t < \infty\}$ of bounded linear operators from X to X is a semigroup of bounded linear operators on X if

(i) $T(0) = I$, I is the identity operator on X.

(ii) $T(t+s) = T(t)T(s)$, for every $t, s \ge 0$ (the semigroup property).

Definition 2.1.2. Let $\{T(t) \mid 0 \le t < \infty\}$ be a semigroup on a Banach space X. The infinitesimal generator , A, of this semigroup is defined by

$$Ax = \lim_{t \to 0+} \frac{1}{t}(T(t)x - x),$$

where x belongs to the domain of A or $D(A) = \{ x \in X \mid \lim\limits_{t \to 0+} \frac{1}{t}(T(t)x - x) \text{ exists} \}$.

Definition 2.1.3. Let $\{T(t) \mid t \ge 0\}$ be a semigroup on a Banach space X. T(t) is uniformly continuous if $\lim\limits_{t \to 0+} \|T(t) - I\|_{L(X)} = 0$, or equivalently, $\lim\limits_{s \to t} \|T(s) - T(t)\|_{L(X)} = 0$.

Theorem 2.1.4. A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.

Proof. See Pazy (1983), pp. 2.

Definition 2.1.5. A semigroup $\{T(t) \mid 0 \le t < \infty\}$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if

$$\lim_{t \to 0+} T(t)x = x, \text{ for every } x \in X.$$

A strongly continuous semigroup of bounded linear operators on X will be called a semigroup of class C_0 or simply a C_0 semigroup.

Theorem 2.1.6. Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup. Then there exists constants $\omega \geq 0$ and $M \geq 1$ such that

$$\|T(t)\|_{L(X)} \leq Me^{\omega t},$$

for $0 \leq t < \infty$.

Proof. See Pazy (1983), pp. 4.

Corollary 2.1.7. If $\{T(t) \mid t \geq 0\}$ is a C_0 semigroup then for every $x \in X$, $t \to T(t)x$ is a continuous function from $[0, \infty)$ into X.

Proof. See Pazy (1983), pp. 4.

Theorem 2.1.8. Let $\{T(t) \mid t \geq 0\}$ be a C_0 semigroup on X and let A be its infinitesimal generator. Then

(a) For $x \in X$, $\lim\limits_{t \to 0+} \frac{1}{h} \int\limits_{0}^{t+h} T(s)x\, ds = T(t)x$.

(b) For $x \in X$, $\int\limits_{0}^{t} T(s)x\, ds \in D(A)$ and

$$A\Big(\int_0^t T(s)x\, ds\Big) = T(t)x - x.$$

(c) For $x \in D(A)$, $T(t)x \in D(A)$, and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

(d) For $x \in D(A)$, $T(t)x - T(s)x = \int\limits_{s}^{t} T(\tau)Ax\, d\tau = \int\limits_{s}^{t} AT(\tau)x\, d\tau$.

Proof. See Pazy (1983), pp. 5.

Corollary 2.1.9. If A is the infinitesimal generator of a C_0 semigroup T(t) on X then D(A), the domain of A, is dense in X and A is a closed linear operator.

Proof. See Pazy, A. (1983), pp. 5-6.

Theorem 2.1.10. Let T(t) and S(t) be C_0 semigroups of bounded linear operators on X with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$, for $t \geq 0$. In other words, a C_0 semigroup T(t), $t \geq 0$ is uniquely determined by its infinitesimal generator.

Proof. See Pazy (1983), pp. 6.

Theorem 2.1.11. Let A be the infinitesimal generator of a C_0 semigroup T(t) on X. If $D(A^n)$ is the domain of A^n, then $\bigcap\limits_{n=1}^{\infty} D(A^n)$ is dense in X.

Proof. See Pazy (1983), pp. 6.

Theorem 2.1.12 (Hille-Yosida Theorem)

A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions T(t), $t \geq 0$ if and only if

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains $[0, \infty)$ and for every $\lambda > 0$,

$$\|R(\lambda; A)\|_{L(X)} \leq 1/\lambda.$$

Proof. See Pazy (1983), pp. 8.

Corollary 2.1.13. A linear operator A is the infinitesimal generator of a C_0 semigroup T(t) satisfying $\|T(t)\|_{L(X)} \leq e^{\omega t}$ for all $t \geq 0$ if and only if

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains the ray $\{ \lambda \mid \text{Im}\lambda = 0, \lambda > \omega \}$ and for such λ

$$\|R(\lambda; A)\|_{L(X)} \leq \frac{1}{\lambda - \omega}.$$

Theorem 2.1.14 A linear operator A with D(A) and R(A) in X is the infinitesimal generator of a C_0 semigroup T(t), $t \geq 0$ on X satisfying $\|T(t)\|_{L(X)} \leq M$ for all $t \geq 0$ (for some $M \geq 1$) if and only if

(i) A is closed, $\overline{D(A)} = X$.

(ii) $\rho(A) \supset (0, \infty)$ and $\| \lambda^n R^n(\lambda, A) \|_{L(X)} \leq M$ for $\lambda > 0$, and $n \in N_0 = \{0, 1, 2, \ldots\}$.

Proof. See Ahmed(1991), pp. 44.

Theorem 2.1.15. Let A be a densely defined linear operator on a Banach space X satisfying the following conditions:

(a1) There exists a $0 < \delta < \pi/2$ such that $\rho(A) \supset \Sigma_\delta \equiv \{ \lambda \in \Re \mid |\arg \lambda| < \pi/2 + \delta \} \cup \{0\}$.

(a2) There exists a constant M>0 such that $\| R(\lambda; A)\|_{L(X)} \leq M/|\lambda|$, for $\lambda \in \Sigma_\delta \setminus \{0\}$.

Then A is the infinitesimal generator of a C_0 semigroup T(t), $t \geq 0$ satisfying

(c1) $\|T(t)\|_{L(X)} \leq K$, for $t \geq 0$ and some constant $K > 0$.

(c2) $T(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda; A)\, d\lambda$,

where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\nu}$ to $\infty e^{i\nu}$ for a fixed $\nu \in (\pi/2, \pi/2 + \delta)$ with the integral converging in the uniform operator topology.

Proof. See Ahmed (1991), pp. 77.

Definition 2.1.16. A C_0 semigroup T(t), $t \geq 0$ on a Banach space X is said to be differentiable if, for each $x \in X$, T(t)x is differentiable for all $t > 0$.

Remark 2.1.17. Note that T(t) is not expected to be differentiable at the origin since that would require its generator to be a bounded operator.

Theorem 2.1.18. If T(t), $t \geq 0$ is a differentiable semigroup with A being its infinitesimal generator then it is differentiable infinitely many times and, for each $n \in N_0$,

(i) $\frac{d^n}{dt^n} T(t) = T^{(n)}(t) = A^n T(t) \in L(X)$, for $t \geq 0$.

(ii) $T^{(n)}(t) = (AT(t/n))^n$, for $t > 0$.

(iii) $T^{(n)}(t)$ is uniformly continuous for $t > 0$.

Proof. See Ahmed (1991), pp. 74.

2.2 Analytic Semigroups

Definition 2.2.1. Let $\Delta = \{ z \in \Re \mid \theta_1 < \arg z < \theta_2 , \theta_1 < 0 < \theta_2 \}$ and suppose $T(z) \in L(X)$ for all $z \in \Delta$. The family $\{T(z) | z \in \Delta \}$ is called an analytic semigroup in Δ if it satisfies the following properties:

(i) $z \to T(z)$ is analytic in Δ (in the sense of uniform operator topology, i. e., for all $z \in \Delta$, $x^* T(z)x$ is analytic in $\Re$, for all $x \in X$, $x^* \in X^*$ such that $\|x\|_X \leq 1$ and $\| x^* \|_{X^*} \leq 1$, and $\|T(z)\|_{L(X)} = \sup_{\|x\| \leq 1} \| T(z)x \|_X$).

(ii) $T(0) = I$ and $\lim_{\substack{z \to 0 \\ z \in \Delta}} T(z)x = x$, for all $x \in X$.

(iii) $T(z_1+z_2) = T(z_1)T(z_2)$, for all $z_1, z_2 \in \Delta$.

A semigroup T(t) will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

Theorem 2.2.2. Let A be the Infinitesimal generator of a uniformly bounded C_0 semigroup T(t), $t \geq 0$, with $0 \in \rho(A)$. Then the following statements are equivalent:

(a) T(t) can be extended to an analytic semigroup from the nonnegative real line to a sector around it, given by $\Delta_\delta \equiv \{ z \mid |\arg z| < \delta \}$ for some $\delta > 0$, and $\|T(z)\|_{L(X)}$ is uniformly bounded on every closed subsector $\Delta_{\delta'} \subset \Delta_\delta , \delta' < \delta$.

(b) There exists a constant $C > 0$ such that, for every $\sigma > 0$ and $\tau \neq 0$,

$$\|R(\sigma + i\tau, A)\|_{L(X)} \leq C/|\tau|.$$

(c) There exists $0 < \delta < \pi/2$, and $M \geq 1$, such that $\rho(A) \supset \Sigma \equiv \{ \lambda \in \Re \mid |\arg \lambda| < \pi/2 + \delta \} \cup \{0\}$

$\| R(\lambda; A)\|_{L(X)} \leq M/|\lambda|$, for $\lambda \in \Sigma \setminus \{0\}$.

(d) T(t) is differentiable for all $t > 0$ and there exists a constant $M_1 > 0$ such that

$$\|AT(t)\|_{L(X)} \leq (M_1/ t) \text{ for } t > 0.$$

Proof. See Ahmed (1991), pp. 82.

2.3 Fractional Powers of Closed Operators

Assumption (F). Let A be a densely defined closed linear operator with D(A) and R(A) in X for which the resolvent set $\rho(A) \supset \Sigma \equiv \{ \lambda \in \Re \mid 0 < \omega < |\arg \lambda| \leq \pi \} \cup V_0$ where V_0 is a neighberhood of zero in $\Re$ and

$$\| R(\lambda; A)\|_{L(X)} \leq M/(1 + |\lambda|), \text{ for } \lambda \in \Sigma. \tag{2.3.1}$$

Definition 2.3.1. Let A be the operator satisfying the assumption (F) and let $\alpha > 0$. Define

$$A^{-\alpha} = \frac{1}{2\pi i} \int_C z^{-\alpha} (A - zI)^{-1} dz \tag{2.3.2}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\vartheta}$ to $\infty e^{i\vartheta}$, $\omega < \vartheta < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z.

The integral (2.3.2) converges in the uniform topology for every $\alpha > 0$ and thus defines a bounded linear operator $A^{-\alpha}$. For $0 < \alpha < 1$ we can deform the path of integration C into the upper and lower sides of the negative real axis and obtain

$$A^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (tI + A)^{-1} dt, \quad 0 < \alpha < 1. \tag{2.3.3}$$

Lemma 2.3.2. Suppose A satisfies the assumption (F) with $0 < \omega < \pi/2$ and let $T(t), t \geq 0$ be the semigroup corresponding to the operator $-A$. Then for every $0 < \alpha < 1$ and $x \in X$ we have

$$A^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)x \, dt, \tag{2.3.4}$$

where $\Gamma(\alpha)$ is the gamma function at α.

Proof. See Ahmed (1991), pp. 91-92.

Remark 2.3.3. Defining $A^{-0} = I$ and using the equations

$$A^{-n} = (1/\Gamma(n)) \int_0^\infty t^{n-1} T(t) dt, \tag{2.3.5}$$

and (2.3.4) one can verify that the equation (2.3.4) holds for all real numbers $\alpha \geq 0$ and not merely for fractions.

Lemma 2.3.4. For $\alpha, \beta \geq 0$, $A^{-(\alpha+\beta)} = A^{-\alpha} A^{-\beta}$.

Proof. See Ahmed (1991), pp. 93.

Lemma 2.3.5. There exists a constant $0 < C < \infty$ such that $\| A^{-\alpha} \|_{L(X)} \leq C$, for all $0 \leq \alpha \leq 1$.

Proof. See Ahmed (1991), pp. 93.

Lemma 2.3.6. For every $x \in X$, $\lim_{\alpha\to 0} A^{-\alpha}x = x$.

Proof. See Ahmed (1991), pp. 94.

Remark 2.3.7. Under the assumption (F), it follows from the above results that $S(t) \equiv A^{-t}$, $t \geq 0$ is itself a C_0 semigroup in X.

Lemma 2.3.8. The operator $A^{-\alpha}$, $\alpha \geq 0$, is one-to-one.

Proof. See Ahmed (1991), pp. 95.

Definition 2.3.9. Suppose that the operator A satisfies the assumption (F) with $0 < \omega < \pi/2$, so that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$, $t \geq 0$. For every $\alpha \geq 0$, we define

$$A^{\alpha} = \begin{cases} (A^{-\alpha})^{-1}, \text{ for } \alpha > 0, \\ I, \text{ for } \alpha = 0. \end{cases} \tag{2.3.6}$$

Clearly by virtue of Lemma 2.3.8, this is a single valued map and its domain $D(A^{\alpha})$ equals the range of $A^{-\alpha}$, i. e., $D(A^{\alpha}) = R(A^{-\alpha})$, for all $\alpha \geq 0$.

Theorem 2.3.10. The operator A^{α}, $0 \leq \alpha \leq 1$, as defined in definition 2.3.9, satisfies the following properties

(i) A^{α} is a closed operator with $D(A^{\alpha}) = R(A^{-\alpha})$.

(ii) $0 < \beta \leq \alpha$ implies $D(A^{\alpha}) \subset D(A^{\beta})$.

(iii) $\overline{D(A^{\alpha})} = X$, for every $\alpha \geq 0$.

(iv) If α, β are real then $A^{\alpha+\beta}x = A^{\alpha}A^{\beta}x$, for $x \in D(A^{\gamma})$, where $\gamma \equiv \max\{\alpha, \beta, \alpha+\beta\}$.

Proof. See Ahmed (1991), pp. 96.

Theorem 2.3.11. Suppose A satisfies the assumption (F) so that $-A$ is the infinitesimal generator of an analytic semigroup. Then, for each α satisfying $0 < \alpha < 1$, the operator A^{α} is given by

$$A^{\alpha}x = \left(\tfrac{\sin \alpha\pi}{\pi}\right) \int_0^{\infty} r^{\alpha-1} A(rI + A)^{-1} x \, dr, \tag{2.3.7}$$

for $x \in D(A)$.

Proof. See Ahmed (1991), pp. 97.

Theorem 2.3.12. Suppose $-A$ is the infinitesimal generator of an analytic semigroup satisfying the assumption (F). Then for $0 < \alpha < 1$ and for every $\sigma > 0$,

$$\| A^{\alpha}x\|_X \leq (1 + M)[\sigma^{\alpha} \| x \|_X + \sigma^{\alpha-1} \| Ax \|_X], \tag{2.3.8}$$

and further,

$$\| A^{\alpha}x \|_X \leq 2(1 + M)\| x \|_X^{1-\alpha} \| Ax \|_X^{\alpha}, \tag{2.3.9}$$

for $x \in D(A)$.

Proof. See Ahmed (1991), pp. 98.

Corollary 2.3.13. Let B be a closed operator with $D(B) \supset D(A^{\alpha})$ for some α satisfying $0 < \alpha \le 1$. Then there exists a constant $K_1 > 0$ such that

$$\|Bx\|_X \le K_1 \|A^{\alpha}x\|_X, \tag{2.3.10}$$

for $x \in D(A^{\alpha})$, and

$$\|Bx\|_X \le K_1(1 + M)[\sigma^{\alpha} \| x \|_X + \sigma^{\alpha-1} \|Ax\|_X], \tag{2.3.11}$$

for $x \in D(A)$ and for every $\sigma > 0$.

Proof. See Ahmed (1991), pp. 99.

Theorem 2.3.14. Suppose B is a closed linear operator with $D(B) \supset D(A)$ and there exists constants $K > 0$ and $\sigma_0 > 0$ such that, for some $0 < \rho < 1$ and every $0 < \sigma \le \sigma_0$,

$$\|Bx\|_X \le K[\sigma^{-\rho} \|x\|_X + \sigma^{1-\rho} \|Ax\|_X], \tag{2.3.12}$$

for all $x \in D(A)$. Then $D(B) \supset D(A^{\alpha})$ for $\rho < \alpha \le 1$.

Proof. See Ahmed (1991), pp. 100.

Remark 2.3.15. For an arbitrary ω appearing in assumption (F), the operator $-A^{\alpha}$, $\alpha \le 1/2$ is the generator of a C_0-semigroup while for $0 < \omega < \pi/2$, $-A^{\alpha}$, $0 < \alpha \le 1$, is the generator of an analytic semigroup.

Proof. See Ahmed (1991), pp. 101.

Theorem 2.3.16. Let -A be the infinitesimal generator of an analytic semigroup $T(t)$, $t \ge 0$ on X and suppose $0 \in \rho(A)$. Then the following results hold

(a) $T(t)X \subset D(A^{\alpha})$, for $t > 0$ and all $\alpha \ge 0$.

(b) For $x \in D(A^{\alpha})$, $T(t)A^{\alpha}x = A^{\alpha}T(t)x$, for all $\alpha \ge 0$.

(c) For each $t > 0$, $A^{\alpha}T(t) \in L(X)$ and

$$\| A^{\alpha}T(t)\|_{L(X)} \le K_{\alpha}t^{-\alpha}e^{-\gamma t}, \tag{2.3.13}$$

$t > 0$, for some constants $K_{\alpha} > 0, \gamma > 0$.

(d) For $0 < \alpha \le 1$ and $x \in D(A^{\alpha})$,

$$\|T(t)x - x \|_X \le C_{\alpha} t^{\alpha} \| A^{\alpha}x \|_X, \tag{2.3.14}$$

for some constant $C_{\alpha} > 0$.

Proof. See Ahmed (1991), pp. 101.

Theorem 2.3.17. (Moment Inequality)

For $0 \le \alpha < \beta \le 1$, there exists a constant $M_{\alpha,\beta}$ such that

$$\| A^{\alpha} x\|_X \leq M_{\alpha,\beta}(\| A^{\beta} x \|_X)^{\alpha/\beta}(\| x \|_X)^{1-(\alpha/\beta)}, \qquad (2.3.15)$$

for all $x \in D(A^{\beta})$.

Proof. See Ahmed (1991), pp. 103.

2.4 Differential Equations on Banach Space

Let X be a Banach space, called the state space and $A \in L(X)$ with D(A) and $R(A) \subset X$ and consider the differential equation on X given by

$$\begin{cases} \frac{dx}{dt} = Ax, t > 0 \\ x(0) = x_0. \end{cases} \qquad (2.4.1)$$

Definition 2.4.1. The Cauchy problem (2.4.1) is said to have a *classical* solution if for each given $x_0 \in D(A)$ there exists a function $x(t) \equiv x(t, x_0)$, $t > 0$ with values in X, satisfying the following properties

(i) x is $C([0, \infty), X) \cap C^1((0, \infty), X)$; that is, x is once continuously differentiable with $\frac{d}{dt}x(t) \in C((0, \infty); X)$.

(ii) $\frac{d}{dt}x(t) = Ax(t)$ for all $t > 0$, and

(iii) $x(0) = x_0$.

Clearly the condition (ii) also implies that $x(t) \in D(A)$ for all $t > 0$.

Theorem 2.4.2. Let A be a densely defined linear operator in X with $\rho(A) \neq \phi$. Then the initial value problem (2.4.1) has a unique classical solution x(t), which is continuously differentiable on $[0, \infty)$, for every initial value $x_0 \in D(A)$ if, and only if, A is the infinitesimal generator of a C_0 semigroup T(t).

Proof. See Pazy (1983), pp. 102.

Theorem 2.4.3. If A is the infinitesimal generator of a differentiable semigroup on X then for every $x_0 \in X$ the initial value problem (2.4.1) has a unique classical solution.

Proof. See Pazy (1983), pp. 104.

Corollary 2.4.4. If A is the infinitesimal generator of an analytic semigroup then for every $x_0 \in X$, the initial value problem (2.4.1) has a unique classical solution.

Proof. See Pazy (1983), pp. 104.

Remark 2.4.5. If A is the infinitesimal generator of a C_0 semigroup which is not differentiable then, in general, if $x_0 \notin D(A)$, the initial value problem (2.4.1) does not have a classical solution.

The function $t \mapsto T(t)x_0$ is then a "generalized solution" of the initial value problem (2.4.1) which we will call a *mild solution*. There are many ways to define generalized solutions of the initial value problem (2.4.1). All lead eventually to $T(t)x_0$. One such way of defining a generalized solution of (2.4.1) is the following: A continuous function x on $[0, \infty)$ is a generalized solution of (2.4.1) if there are $x_n \in D(A)$ such that $x_n \to x(0)$ as $n \to \infty$ and $T(t)x_n \to x(t)$ uniformly on bounded intervals. It is obvious that the generalized solution thus defined is independent of the sequence (x_n), is unique and if $x(0) \in D(A)$ it gives the solution of (2.4.1). Clearly with this definition of generalized solution, (2.4.1) has a generalized solution for every $x_0 \in X$ and this generalized solution is $T(t)x_0$.

Definition 2.4.6. If A is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X then for every $x_0 \in X$, the function $x(t) \equiv T(t)x_0$, $t \geq 0$ is called the mild solution of the initial value problem (2.4.1).

Theorem 2.4.7. Let A be the generator of a C_0 semigroup $T(t)$, $t \geq 0$, on X. Then

(i) For $x \in D(A^n)$, $n \in \subseteq$, $T(t)x = \sum_{0 \leq k \leq n-1} (t^k / k!)A^k x + \int_0^t [(t - \eta)^{n-1} / (n-1)!]\, T(\eta)(A^n x)\, d\eta$

for $t \geq 0$.

(ii) On any finite interval every mild solution of the Cauchy problem (2.4.1) can beapproximated to any degree of accuracy by a C^∞- function admitting the ∞ -series representation,

$$\sum_{0 \leq k < \infty} (t^k / k!)A^k \eta,$$

for a suitable $\eta \in X$.

Proof. See Ahmed (1991), pp. 150.

Nonhomogeneous Cauchy Problem

Consider the Cauchy problem,

$$\begin{cases} \dfrac{dx}{dt} = Ax + f(t), t > 0, \\ x(0) = x_0. \end{cases} \tag{2.4.2}$$

where $x_0 \in X$ and $f \in L_1([0, \infty); X)$.

Definition 2.4.8. (Classical Solution)

A function $x : [0, a) \to X$ is said to be a classical solution of the Cauchy problem (2.4.2) if

(i) $x \in C([0, a); X) \cap C^1((0, a); X)$.

(ii) $x(t) \in D(A)$ for $t \in (0, a)$.

(iii) x satisfies (2.4.2) on (0, a).

Notation: For $M \geq 1$ and $\omega \in \nabla$, let $G(M, \omega)$ denote the class of infinitesimal generators of C_0 semigroups $\{T(t) \mid t \geq 0\}$ of bounded linear operators on X such that $\|T(t)\|_{L(X)} \leq M \exp(\omega t)$, $t \geq 0$.

Lemma 2.4.9. If the operator $A \in G(M, \omega)$ with $\{T(t) \mid t \geq 0\}$ being the corresponding semigroup and if the Cauchy problem (2.4.2) has a classical solution x in the sense of definition 2.4.8, then x is uniquely defined by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds, \quad t > 0. \tag{2.4.3}$$

Proof. See Ahmed (1991), pp. 152.

Definition 2.4.10. (Mild Solution)

A function $x \in C(I, X)$, for any finite interval $I \equiv [0, a]$, is said to be a mild solution of the Cauchy problem (2.4.2) corresponding to the initial state $x_0 \in X$ and the input $f \in L_1(I, X)$ if x is given by the expression (2.4.3) for $t \in I$.

Theorem 2.4.11. Consider the Cauchy problem (2.4.2) with $x_0 \in D(A)$ and $f \in L_1([0, a]; X) \cap C((0, a); X)$ and suppose that $A \in G(M, \omega)$ with $\{T(t) \mid t \geq 0\}$ being the corresponding semigroup, and let

$$x(t) = T(t)x_0 + z(t), \quad t \in [0, a),$$

where $z(t) \equiv \int_0^t T(t-s)f(s)ds, \ t \in I \equiv [0, a]$, be the associated mild solution. Then, in order that x be a classical solution, it is necessary and sufficient that any one of the following conditions hold

(i) $z \in C^1((0, a); X)$.

(ii) $z(t) \in D(A)$ for $t \in (0, a)$ and $Az(t) \in C((0, a); X)$.

Proof. See Ahmed (1991), pp. 153.

Corollary 2.4.12 Suppose $A \in G(M, \omega)$ with $\{T(t) \mid t \geq 0\}$ being the corresponding semigroup. If $f \in C^1([0, a]; X)$ and $x_0 \in D(A)$, then the Cauchy problem (2.4.2) has a unique (classical) solution.

Proof. See Ahmed (1991), pp. 155.

Corollary 2.4.13. Let $A \in G(M, \omega)$ with $\{T(t) \mid t \geq 0\}$ being the corresponding semigroup. Then for every $x_0 \in D(A)$ and $f \in L_1([0, a]; X)$ satisfying (a) $f(t) \in D(A)$ and (b) $Af \in L_1([0, a]; X)$, the Cauchy problem (2.4.2) has a unique (classical) solution.

Theorem 2.4.14. Let $A \in G(M, \omega)$ with $\{T(t) \mid t \geq 0\}$ being the corresponding semigroup and $f \in L_1([0, a]; X)$ and $x_0 \in X$. Then on any subinterval $[0, b]$, $b < a$, the mild solution x of the initial value problem (2.4.2) given by (2.4.3), is the uniform limit of classical solutions.

Proof. See Ahmed (1991), pp. 155.

Let I be an interval. A function $f : I \to X$ is Hölder continuous with exponent ϑ, $0 < \vartheta < 1$ on I if there is a constant L such that

$$\|f(t) - f(s)\|_X \leq L|t - s|^{\vartheta},$$

for s, $t \in I$. It is locally Hölder continuous if every $t \in I$ has a neighberhood in which f is Hölder continuous. We denote the family of all Hölder continuous functions with exponent ϑ on I by $C^{\vartheta}(I; X)$.

Theorem 2.4.15. Let A be the infinitesimal generator of an analytic semigroup T(t) and $f \in L_p([0, T]; X)$ with $1 < p < \infty$. If x is the mild solution of the problem (2.4.2) then x is Hölder continuous with exponent $(p - 1)/p$ on $[\varepsilon, T]$, for every $\varepsilon > 0$. If moreover $x_0 \in D(A)$ then x is Hölder continuous with the same exponent on [0, T].

Proof. See Pazy (1983), pp. 110.

Theorem 2.4.16. Let A be the infinitesimal generator of an analytic semigroupT(t). Let $f \in L_1([0, T]; X)$ and assume that for every $0 < t < T$, there is a $\delta_t > 0$ and a continuous real valued function $W_t(\tau) : [0, \infty) \to [0, \infty)$ such that

$$\|f(t) - f(s)\|_X \leq W_t(|t-s|)$$

and

$$\int_0^{\delta_t} \frac{W_t(\tau)}{\tau} d\tau < \infty.$$

Then for every $x_0 \in X$ the mild solution of (2.4.2) is a classical solution.

Proof. See Pazy (1983), pp. 111.

Corollary 2.4.17. Let A be the infinitesimal generator of an analytic semigroup T(t). If $f \in L_1([0, T]; X)$ is locally Hölder continuous on (0, T] then for every $x_0 \in X$ the initial value problem (2.4.2) has a unique classical solution x.

Lemma 2.4.18. Let A be the infinitesimal generator of an analytic semigroup T(t) and let $f \in C^{\vartheta}([0, T]; X)$. If $v_1(t) = \int_0^t T(t-s)(f(s) - f(t))ds$ then $v_1(t) \in D(A)$ for $0 \leq t \leq T$ and $Av_1(t) \in C^{\vartheta}([0, T]; X)$.

Proof. See Pazy (1983), pp. 113.

Theorem 2.4.19. Let A be the infinitesimal generator of an analytic semigroup T(t) and let $f \in C^{\vartheta}([0, T]; X)$. If x is the solution of the initial value problem (2.4.2) on [0, T] then

(i) For every $\delta > 0$, $Ax \in C^{\vartheta}([\delta, T]; X)$ and $\frac{dx}{dt} \in C^{\vartheta}([\delta, T]; X)$.

(ii) If $x_0 \in D(A)$ then Ax and $\frac{dx}{dt}$ are continuous on [0, T].

(iii) If $x_0 = 0$ and f(0) = 0 then Ax, $\frac{dx}{dt} \in C^{\vartheta}([\delta, T]; X)$.

Proof. See Pazy (1983), pp. 114.

Theorem 2.4.20. Let A be the infinitesimal generator of an analytic semigroup T(t) on X and let $0 \in \rho(A)$. If f(s) is continuous, $f(s) \in D((-A)^{\alpha})$, $0 < \alpha \leq 1$ and $\|(-A)^{\alpha} f(s)\|_X$ is bounded, then for every $x_0 \in X$ the mild solution of (2.4.2) is a classical solution.

Proof. See Pazy (1983), pp 115.

Semilinear Evolution Equations

Consider the semilinear evolution equation

$$\begin{cases} \frac{dx}{dt} + Ax = f(t, x), t > 0, \\ \quad x(0) = x_0, \end{cases} \tag{2.4.4}$$

on a Banach space X.

Definition 2.4.21. A function $x \in C(I, X)$, $I \equiv [0, a]$, is said to be a mild solution of (2.4.4) if x satisfies the integral equation

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s))\, ds, t \in I. \tag{2.4.5}$$

Theorem 2.4.22. Let –A be the infinitesimal generator of a C_0 semigroup on a Banach space X and $t \mapsto f(t, \xi)$ be a continuous X-valued function for each $\xi \in X$, and suppose there exists a positive constant K such that for all $\xi, \eta \in X$,

$$\|f(t, \xi) - f(t, \eta)\|_X \leq K\|\xi - \eta\|_X, \text{ for all } t \in I.$$

Then, for every $x_0 \in X$, the system (2.4.4) has a unique mild solution $x \in C(I, X)$. Furthermore, $x_0 \mapsto x$ is Lipschitz continuous from X to C(I, X).

Proof. See Ahmed (1991), pp. 168.

Corollary 2.4.23. If A and f satisfy the assumptions of Theorem 2.4.22 and $v \in C(I, X)$ then the integral equation $x(t) = v(t) + \int_0^t T(t-s)f(s, x(s))\, ds, t \in I$, has a unique solution $x \in C(I, X)$.

Theorem 2.4.24. Let –A be the infinitesimal generator of a C_0 semigroup T(t), $t \geq 0$ on X and $f : [0, \infty) \times X \to X$ continuous and locally Lipschitz in the sense that, for every r > 0 and $t_1 > 0$ there exists a constant $K \equiv K(t_1, r)$ such that

$$\|f(t, \xi) - f(t, \eta)\|_X \leq K\|\xi - \eta\|_X,$$

for all $t \in [0, t_1]$ and $\xi, \eta \in B_r \equiv \{\zeta \in X \mid \|\zeta\| \leq r\}$.

Then for every $x_0 \in X$, there exists a $t_m = t_{max}(x_0) \le \infty$ such that the Cauchy problem (2.4.4) has a unique mild solution $x \in C([0, t_m); X)$. Further if $t_m < \infty$ then $\lim\limits_{t \to t_m} \| x(t) \|_X = \infty$.

2.5 Gronwall's Lemma

Lemma 2.5.1. Let f, g : $[t_0, T_0] \to \nabla$ be continuous functions with g nondecreasing, and which, for fixed $c > 0$, satisfy the equality

$$f(t) \le g(t) + c \int_{t_0}^{t} f(s)\, ds \text{, for all } t \in [t_0, T_0].$$

Then $f(t) \le g(t)\, e^{c(t-t_0)}$ for all $t \in [t_0, T_0]$.

Proof. See Zeidler (1984), pp. 82.

Lemma 2.5.2.

Let $0 \le \alpha < 1$ and suppose that $g \in L_1(0, T)$ is nonnegative a. e.. If $w \in L_1(0, T)$ satisfies the integral inequality

$$w(t) \le g(t) + K \int_{t_0}^{t} (t - \tau)^{-\alpha}\, w(\tau)\, d\tau,$$

for almost all $t \in [0, T]$ and for some $K > 0$ then

$$w(t) \le g(t) + K \int_{t_0}^{t} (t - \tau)^{-\alpha}\, m_\alpha(K(t - \tau)^{1-\alpha}) g(\tau)\, d\tau,$$

for almost all $t \in (0, T)$ where

$$m_\alpha(\xi) = \sum_{k=1}^{\infty} \frac{[\Gamma(1\text{-}\alpha)]^k \xi^{k-1}}{\Gamma(k(1-\alpha))}, \xi \in \nabla, 0 \le \alpha < 1.$$

Proof. See Amann (1978).

Corollary 2.5.3. Suppose $w \in L_1(0, T)$ satisfies

$$w(t) \le c_0 t^{-\beta} + c_1 \int_{0}^{t} (t - \tau)^{-\alpha}\, w(\tau)\, d\tau,$$

for almost all $t \in (0, T)$, where c_0, c_1 are nonnegative constants and $0 \le \alpha, \beta < 1$ then there exists a constant $C \equiv C(\alpha, c_1, T)$ such that

$$w(t) \le c_0 C\, t^{-\beta}, \text{ a. e. } t \in (0, T).$$

Proof. See Amann(1978).

Lemma 2.5.4. (Abstract Gronwall's Lemma)

Let $A : X \to X$ be a continuous linear positive operator on the ordered Banach space X with spectral radius $r(A) < 1$. Let $x, y, g \in X$. Then $x \le g + Ax$ and $y = g + Ay$ always imply $x \le y$.

Proof. See Zeidler (1984), pp. 281.

Corollary 2.5.5. Let g, h, $x \in C([a, b])$ with $h \geq 0$ on [a, b]. Let $H(t) = \int_0^t h(s)ds$, it follows that if

$$x(t) \leq g(t) + \int_0^t h(s)x(s)ds,$$

for all $t \in [a, b]$,

then

$$x(t) \leq g(t) + \int_0^t g(s)h(s)e^{H(t)-H(s)}ds,$$

for all $t \in [a, b]$.

In particular, if g is monotone increasing and $h(s) \equiv c$ with $c > 0$, then we obtain

$$x(t) \leq g(t) \exp(c(t-a)),$$

for all $t \in [a, b]$.

Proof. See Zeidler (1984), pp. 282.

Lemma 2.5.6. (Gronwall's Lemma with Time Lag)

Suppose $x \in C \equiv C([-r, T]; X)$ satisfies the following inequality

$$\begin{cases} \| x(t) \| \leq a + b\int_0^t \| x(s) \| ds + c\int_0^t \| x_s \|_C \, ds, \ t \in [0, T], \\ x(t) = \varphi(t), \ t \in [-r, 0], \end{cases}$$

where $\varphi \in C$ and $a, b, c \geq 0$ are constants and $\| x_s \|_C = \sup_{-r \leq \theta \leq 0} \| x(s + \theta) \|_X$. Then

$$\| x(t) \|_X \leq (a + cT\| \varphi \|_C)e^{(b+c)t}.$$

Proof. See Xiang and Kuang (2000).

Chapter III

Semilinear Integrodifferential Equations and Analytic Semigroups

In this chapter, we study existence of mild solutions for a class of semilinear integrodifferential equations with finite delay. We discuss this problem in several kinds of situations. The theory of analytic semigroups, and the Banach contraction mapping theorem are important tools to prove local existence and uniqueness of mild solutions. We impose an a priori estimate condition to achieve extension of local mild solutions. A global existence theorem is proved. We also study the regularity of mild solutions and continuous dependence. The existence problem of mild solutions for a system with infinite delay is investigated.

Let X be a Banach space (over R or C), and $r \geq 0$, $T > 0$, $0 < \alpha < 1$ be given. Let L(X) denote the Banach space of linear and bounded operators on X with the supremum norm. For an infinitesimal generator –A of an analytic semigroup T(t), t ≥ 0, we can define a fractional power operator A^{α} and $D(A^{\alpha})$ is the Banach space endowed with the graph norm defined by $|||x||| = \|A^{\alpha}x\|_X + \|x\|_X$, $x \in D(A^{\alpha})$. By the invertibility of A^{α}, the graph norm $|||\cdot|||$ is equivalent to the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|_X$. Throughout this thesis, we denote by X_{α}, the Banach space $D(A^{\alpha})$ equipped with the norm $\|\cdot\|_{\alpha}$. Here are assumptions that are used to prove the existence of solutions and other related properties.

Assumptions

(A) $-A$ is the infinitesimal generator of an analytic semigroup T(t) on X satisfying $\|T(t)\|_{L(X)} \leq M$ for all $t \geq 0$, and $0 \in \rho(-A)$.

(F1) The function $f : X_{\alpha} \to X$ is locally Lipschitz continuous in x $\in X_{\alpha}$, i. e., for each $\rho > 0$ there exists a constant $K_1(\rho) > 0$ such that

$$\|f(x_1) - f(x_2)\|_X \leq K_1(\rho)\|x_1 - x_2\|_{\alpha},$$

for all $x_1, x_2 \in X_{\alpha}$ such that $\|x_1\|_{\alpha} \leq \rho$ and $\|x_2\|_{\alpha} \leq \rho$.

(G1) The function $g : X_{\alpha} \to X$ is locally Lipschitz continuous in x $\in X_{\alpha}$, i. e., for each $\rho > 0$ there exists a constant $K_2(\rho) > 0$ such that

$$||g(x_1) - g(x_2)||_X \leq K_2(\rho)||x_1 - x_2||_\alpha,$$

for all $x_1, x_2 \in X_\alpha$ such that $||x_1||_\alpha \leq \rho$ and $||x_2||_\alpha \leq \rho$.

(F2) The function $f : [0, T] \times X_\alpha \to X$ satisfies

(i) $f(\bullet, x)$ is continuous on $[0, T]$, for each $x \in X_\alpha$.

(ii) $f(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [0, T]$, i. e., for each $t \in [0, T]$ and each $\rho > 0$ there exists a constant $K_1 = K_1(t, \rho) > 0$ such that

$$||f(s, x_1) - f(s, x_2)||_X \leq K_1|| x_1 - x_2||_\alpha,$$

for all $s \in [0, t]$ and all $x_1, x_2 \in X_\alpha$ such that $|| x_1||_\alpha \leq \rho$, $|| x_2||_\alpha \leq \rho$.

(G2) The function $g : [-r, T] \times X_\alpha \to X$ satisfies

(i) $g(\bullet, x)$ is continuous on $[-r, T]$, for each $x \in X_\alpha$.

(ii) $g(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [-r, T]$.

(F3) The function $f : [0, T] \times X_\alpha \to X$ satisfies

(i) $f(\bullet, x)$ is measurable on $[0, T]$, for each $x \in X_\alpha$.

(ii) $f(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [0, T]$.

(iii) f maps every bounded set in $[0, T] \times X_\alpha$ to a bounded set in X.

(G3) The function $g : [-r, T] \times X_\alpha \to X$ satisfies

(i) $g(\bullet, x)$ is measurable on $[-r, T]$, for each $x \in X_\alpha$.

(ii) $g(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [-r, T]$.

(iii) g maps every bounded set in $[-r, T] \times X_\alpha$ to a bounded set in X.

(F4) The function $f : [0, T] \times X_\alpha \to X$ satisfies

(i) $f(\bullet, x)$ is locally Hölder continuous on $[0, T]$, for each $x \in X_\alpha$, i. e., for each $x_0 \in X_\alpha$ and each $t \in [0, T]$, there exists a neighberhood $V \subset [0, T] \times X_\alpha$ of (t, x_0) and a constant L such that

$$||f(s_1, x) - f(s_2, x)||_X \leq L| s_1 - s_2 |^\nu,$$

for all $s_1, s_2 \in [0, t]$ such that $(s_1, x), (s_2, x) \in V$, for some exponent $\nu \in (0, 1)$.

(ii) $f(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [0, T]$.

(G4) The function $g : [-r, T] \times X_\alpha \to X$ satisfies

(i) $g(\bullet, x)$ is locally Hölder continuous on $[-r, T]$, for each $x \in X_\alpha$.

(ii) $g(t, \bullet)$ is locally Lipschitz continuous on X_α, for each $t \in [-r, T]$.

(F5) The function $f : X_\alpha \to X$ satisfies a growth condition, i. e., there exists a constant $K_1 > 0$ such that

$$\|f(x)\|_X \leq K_1(1 + \|x\|_\alpha),$$

for all $x \in X_\alpha$.

(G5) The function $g: X_\alpha \to X$ satisfies a growth condition, i. e., there exists a constant $K_2 > 0$ such that

$$\|g(x)\|_X \leq K_2(1 + \|x\|_\alpha),$$

for all $x \in X_\alpha$.

(F6) Suppose there exists a Banach space E with $X_\alpha \subset E \subset X$ and a constant $\lambda \in [1, \frac{1}{\alpha})$ such that for every $\rho > 0$ there exists a constant $c(\rho) > 0$ such that

$$\|f(x)\|_X \leq c(\rho)(1 + \| x \|_\alpha^\lambda),$$

for every $x \in X_\alpha$ satisfying $\| x \|_E \leq \rho$.

(G6) Suppose there exists a Banach space E with $X_\alpha \subset E \subset X$ and a constant $\lambda \in [1, \frac{1}{\alpha})$ such that for every $\rho > 0$ there exists a constant $d(\rho) > 0$ such that

$$\|g(x)\|_X \leq d(\rho)(1 + \| x \|_\alpha^\lambda),$$

for every $x \in X_\alpha$ satisfying $\| x \|_E \leq \rho$.

(H1) $h \in L_1([0, T + r]; L(X))$.

(H2) $h \in L_p([0, T + r]; L(X))$, for $1 < p < \infty$.

3.1 Local Existence of Mild Solutions

We consider semilinear integrodifferential equations as follows:

$$\begin{cases} \frac{dx}{dt} + Ax(t) = f(x(t)) + \int_{-r}^{t} h(t-s)g(x(s))ds, t \in (0,T], \\ x(t) = \varphi(t), t \in [-r,0]. \end{cases} \tag{3.1.1}$$

Definition 3.1.1. A function $x \in C([-r, T]; X_\alpha) \cap C^1((0, T); X)$ is called a *classical* solution of the system (3.1.1) if it satisfies the system (3.1.1) with $\varphi \in C([-r, 0]; X_\alpha)$.

Definition 3.1.2. A function $x \in C([-r, a]; X_\alpha)$, $a \in [0,T]$, is called a *mild* solution of the system (3.1.1) if it satisfies the integral equation (3.1.2)

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(x(s))ds + \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(x(\theta))d\theta]ds, t \in [0,a], \\ \varphi(t), t \in [-r,0]. \end{cases} \tag{3.1.2}$$

In the following we deal with the problem of local existence which is one of main parts of our thesis. Analytic semigroups, locally Lipschitz condition, and the Banach contraction mapping theorem are important tools to solve this problem. An a- priori estimate is a very important condition to prove extension theorem. To obtain global existence of mild solutions, we impose a nearly linear growth condition and a super linear growth condition. We consider existence problems in several kinds of situations.

Let $C \equiv C([0,\ T]; X_\alpha)$ denote the Banach space of all continuous X_α-valued functions defined on [0, T], with the supremum norm. For a fixed $\varphi \in C([-r,0]; X_\alpha)$, let C_φ denote $\{x \in C \mid x(0) = \varphi(0)\}$. Then C_φ is a nonempty closed convex subset of C. We denote $\int_0^{T+r} \| h(\theta) \|_{L(X)}\, d\theta$ by $\bar{h}$.

Lemma 3.1.3. Assume that (A), (F1), (G1), and (H1) hold. For any $\varphi \in C([-r, 0]; X_\alpha)$, define a mapping G on C_φ by

$$(Gx)(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(x(s))ds + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta))d\theta]ds,\ t \in [0,T],$$

where $x \in C_\varphi$ and $\tilde{x}(t) = \begin{cases} x(t), t \in [0,T], \\ \varphi(t), t \in [-r,0]. \end{cases}$ (3.1.3)

Then $G : C_\varphi \to C_\varphi$.

Proof. Let $x \in C_\varphi$. We show that $Gx \in C_\varphi$. Clearly, $(Gx)(0) = \varphi(0)$.

First, we show that $\sup_{s\in[0,T]} \|f(x(s))\|_X$ and $\sup_{s\in[-r,T]} \|g(\tilde{x}(s))\|_X$ are bounded, then we will show that Gx is continuous on [0, T].

By definition of $\tilde{x}$, $\tilde{x}$ is a continuous X_α-valued function on [-r, T], then there exists a constant $\rho > 0$ such that $\| \tilde{x}(s) \|_\alpha \le \rho$, for all $s \in$ [-r, T].

Since f is locally Lipschitz on X_α, $\| x(s) \|_\alpha \le \rho$ for all $s \in$ [0, T] and $\| \varphi(0) \|_\alpha \le \rho$ then

$$\begin{aligned}
\sup_{s\in[0,T]} \|f(x(s))\|_X &\le \sup_{s\in[0,T]} \| f(x(s)) - f(x(0)) \|_X + \|f(x(0))\|_X \\
&\le K_1(\rho)\left(\sup_{s\in[0,T]} \| x(s) - \varphi(0) \|_\alpha \right) + \|f(\varphi(0))\|_X \\
&\le K_1(\rho)(\sup_{s\in[0,T]} \| x(s) \|_\alpha + \| \varphi(0) \|_\alpha) + \|f(\varphi(0))\|_X \\
&\le 2\rho\, K_1(\rho) + \| f(\varphi(0)) \|_X \equiv \overline{M}. \qquad (3.1.4)
\end{aligned}$$

Note that $\overline{M}$ depends only on ρ and φ.

Since g is locally Lipschitz on X_α and $\| \varphi(s) \|_\alpha \le \rho$ for all $s \in [-r, 0]$, then there exists a constant $K_2(\rho)$ such that

$$\| g(\varphi(s)) - g(\varphi(0)) \|_X \le K_2(\rho) \| \varphi(s) - \varphi(0) \|_\alpha ,$$

$$\| g(x(s)) - g(x(0)) \|_X \le K_2(\rho) \| x(s) - x(0) \|_\alpha .$$

Then

$$\begin{aligned}
\sup_{s\in[-r,T]} \|g(\tilde{x}(s))\|_X &\le \sup_{-r\le s\le 0} \| g(\varphi(s)) \|_X + \sup_{0\le s\le T} \| g(x(s)) \|_X \\
&\le \sup_{-r\le s\le 0} (\| g(\varphi(s)) - g(\varphi(0)) \|_X) + \| g(\varphi(0)) \|_X \\
&\quad + \sup_{0\le s\le T} (\| g(x(s)) - g(x(0)) \|_X) + \| g(x(0)) \|_X \\
&\le K_2(\rho) \left(\sup_{-r\le s\le 0} \| \varphi(s) - \varphi(0) \|_\alpha \right) + \| g(\varphi(0)) \|_X \\
&\quad + K_2(\rho) \left(\sup_{0\le s\le T} \| x(s) - \varphi(0) \|_\alpha \right) + \|g(\varphi(0))\|_X \\
&\le K_2(\rho) \left((\sup_{-r\le s\le 0} \| \varphi(s) \|_\alpha) + \| \varphi(0) \|_\alpha \right) + \| g(\varphi(0)) \|_X \\
&\quad + K_2(\rho) \left((\sup_{0\le s\le T} \| x(s) \|_\alpha) + \| \varphi(0) \|_\alpha \right) + \|g(\varphi(0))\|_X \\
&\le 4\rho K_2(\rho) + 2\|g(\varphi(0))\|_X \equiv \overline{N}. \qquad (3.1.5)
\end{aligned}$$

Note that $\overline{N}$ depends only on ρ and φ.

We now show that Gx is continuous on [0, T).

Let $t \in [0, T)$ and let ξ be such that $0 \le t < t + \xi < T$. Then

$$\begin{aligned}
&\| (Gx)(t+\xi) - (Gx)(t) \|_\alpha \\
&\le \| T(t+\xi)\varphi(0) - T(t)\varphi(0) \|_\alpha \\
&\quad + \| \int_0^{t+\xi} T(t+\xi-s) f(x(s)) ds - \int_0^t T(t-s) f(x(s)) ds \|_\alpha \\
&\quad + \| \int_0^{t+\xi} T(t+\xi-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta)) d\theta] ds - \int_0^t T(t-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta)) d\theta] ds \|_\alpha \\
&\le \| (T(\xi) - I) T(t) A^\alpha \varphi(0) \|_X \\
&\quad + \| \int_0^t (T(t+\xi-s) - T(t-s)) f(x(s)) ds \|_\alpha + \int_t^{t+\xi} \| T(t+\xi-s) f(x(s)) \|_X \, ds \\
&\quad + \| \int_0^t (T(t+\xi-s) - T(t-s))[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta)) d\theta] ds \|_\alpha
\end{aligned}$$

$$
\begin{aligned}
&+\int_t^{t+\xi} \| T(t+\xi-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta] \|_\alpha \, ds \\
\leq\ & \| (T(\xi)-I)T(t)A^{\alpha}\varphi(0) \|_X \\
&+ \| (T(\xi)-I)[\int_0^t T(t-s)f(x(s))ds] \|_\alpha \\
&+ K_\alpha \int_t^{t+\xi} (t+\xi-s)^{-\alpha} \| f(x(s)) \|_X \, ds \\
&+ \| (T(\xi)-I)\int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta] ds \|_\alpha
\end{aligned}
$$

$$+K_\alpha \int_t^{t+\xi} (t+\xi-s)^{-\alpha}\Big[\int_{-r}^{s} \|h(s-\theta)\|_{L(X)}\|g(\tilde{x}(\theta))\|_X \, d\theta\Big]ds$$

$$\le \|(T(\xi)-I)T(t)A^\alpha\varphi(0)\|_X$$

$$+\Big\|(T(\xi)-I)A^\alpha\Big[\int_0^t T(t-s)f(x(s))ds\Big]\Big\|_X$$

$$+K_\alpha\overline{M}\,\frac{\xi^{1-\alpha}}{1-\alpha}$$

$$+\Big\|(T(\xi)-I)A^\alpha\Big[\int_0^t T(t-s)\Big[\int_{-r}^{s} h(s-\theta)g(x(\theta))d\theta\Big]ds\Big]\Big\|_X$$

$$+K_\alpha\overline{N}\,\overline{h}\,\frac{\xi^{1-\alpha}}{1-\alpha}.$$

Since $\varphi(0) \in X_\alpha$, $T(t)A^\alpha\varphi(0) \in X$ and $T(t)$ is strongly continuous then $\|(T(\xi)-I)T(t)A^\alpha\varphi(0)\|_X \to 0$ as $\xi \to 0^+$.

Since $f(x(s)) \in X$, $T(t) : X \to X_\alpha$ is strongly continuous and $A^\alpha : X_\alpha \to X$ then

$\|(T(\xi)-I)A^\alpha[\int_0^t T(t-s)f(x(s))ds]\|_X \to 0^+$.

Since $h(s-\theta) \in L(X)$, $g : X_\alpha \to X$ and $x(s) \in X_\alpha$ then $\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta \in X$.

Since $T(t): X \to X_\alpha$ then $\int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta]ds \in X_\alpha$, and so

$A^\alpha \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta]ds \in X$. Since $T(t)$ is strongly continuous then

$\|(T(\xi)-I)[A^\alpha \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta]ds]\|_X \to 0$ as $\xi \to 0^+$.

Hence $\|(Gx)(t+\xi)-(Gx)(t)\|_\alpha \to 0$ as $\xi \to 0^+$.

By a similar argument, it follows that $\|(Gx)(T-\xi)-(Gx)(T)\|_\alpha \to 0$ as $\xi \to 0^+$.

Then Gx is continuous on [0, T].

Hence $Gx \in C_\varphi$. The proof is complete. □

Theorem 3.1.4. (Local Existence Theorem) Assume that (A), (F1), (G1), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then there exists a positive number t_1 such that the system (3.1.1) has a unique mild solution on $[-r, t_1]$.

Proof. Let $t_1 \in (0,T]$. Set $B = \{x \in C_\varphi \mid \|x(t) - \varphi(0)\|_\alpha \le 1, t \in [0,t_1]\}$.

Define a mapping G on B by

$$(Gx)(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(x(s))ds + \int_0^t T(t-s)\Big[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta\Big]ds,\ t \in [0,t_1],$$

where $x \in B$ and $\tilde{x}(t) = \begin{cases} x(t), t \in [0,t_1], \\ \varphi(t), t \in [-r,0]. \end{cases}$ (3.1.6)

We will show there exists a $t_1 > 0$ such that G maps B to B and G is a contraction mapping. Then by the Contraction mapping theorem, G has a unique fixed point in B. This means that the system (3.1.1) has a unique local mild solution.

Since $\beta > \alpha$ then $X_\beta \subset X_\alpha$, let c_1 be a constant such that $\| x \|_\alpha \le c_1 \| x \|_\beta$, for all $x \in X_\beta$.
Let $\rho \equiv 1 + c_1 \| \varphi(0) \|_\beta$.
As in Lemma 3.1.3, there exists $\overline{M}, \overline{N}$ depending only on ρ and φ such that

$$\sup_{s \in [0,T]} \|f(x(s))\|_X \le \overline{M},$$

$$\sup_{s \in [-r,T]} \|g(x(s))\|_X \le \overline{N},$$

provided $x \in B$. Let $K_1(\rho)$ and $K_2(\rho)$ be Lipschitz constants of f and g respectively. By the properties 2.3.16 (c), (d) of analytic semigroups, since $A^\alpha \varphi(0) \in X_{\beta-\alpha}$, there exist constants $C_{\beta-\alpha} > 0$ and $K_\alpha > 0$ such that

$$\|T(t)\varphi(0) - \varphi(0)\|_\alpha \le C_{\beta-\alpha} t^{\beta-\alpha} \| \varphi(0) \|_\beta ,$$

and

$$\| A^\alpha T(t) \|_{L(X)} \le K_\alpha t^{-\alpha},$$

for all $t > 0$.

Set $\overline{K} = \overline{M} + K_1(\rho) + (\overline{N} + K_2(\rho))\overline{h}$. Fix $L \in (0, 1)$.

Choose $t_1 = \min\{1, T, \left(\frac{1}{L}(C_{\beta-\alpha}\rho + \frac{K_\alpha \overline{K}}{\beta-\alpha}) \right)^{\frac{-1}{\beta-\alpha}} \}$.

At first, we show that $G: B \to B$. Let $x \in B$.

Then $\| x(t) \|_\alpha \le 1 + \| \varphi(0) \|_\alpha \le 1 + c_1 \| \varphi(0) \|_\beta = \rho$, for all $t \in [0, t_1]$.

By Lemma 3.1.3, $G : C_\varphi \to C_\varphi$. So $Gx \in C_\varphi$.

For $t \in [0, t_1]$,

$$\begin{aligned}
&\|(Gx)(t) - \varphi(0)\|_\alpha \\
&\qquad \le \|T(t)\varphi(0) - \varphi(0)\|_\alpha + \int_0^t \| T(t-s) f(x(s)) \|_\alpha \, ds \\
&\qquad\qquad + \int_0^t \| T(t-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta)) d\theta] \|_\alpha \, ds \\
&\qquad \le C_{\beta-\alpha} t^{\beta-\alpha} \| \varphi(0) \|_\beta + K_\alpha \int_0^t (t-s)^{-\alpha} \| f(x(s)) \|_X \, ds \\
&\qquad\qquad + K_\alpha \int_0^t (t-s)^{-\alpha} [\int_{-r}^s \| h(s-\theta) \|_{L(X)} \| g(\tilde{x}(\theta)) \|_X \, d\theta] ds \\
&\qquad \le C_{\beta-\alpha} t^{\beta-\alpha} \| \varphi(0) \|_\beta + K_\alpha \overline{M} \frac{t^{1-\alpha}}{1-\alpha} + K_\alpha \overline{N} (\int_0^{T+r} \| h(\theta) \|_{L(X)} \, d\theta) \frac{t^{1-\alpha}}{1-\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\le C_{\beta-\alpha} t_1^{\beta-\alpha} \| \varphi(0) \|_\beta + K_\alpha(\overline{M} + \overline{N}\,\overline{h}) \frac{t_1^{1-\alpha}}{1-\alpha} \\
&\le C_{\beta-\alpha} t_1^{\beta-\alpha} \| \varphi(0) \|_\beta + K_\alpha \overline{K} \frac{t_1^{1-\alpha}}{1-\alpha} \\
&\le C_{\beta-\alpha} t_1^{\beta-\alpha} \| \varphi(0) \|_\beta + K_\alpha \overline{K} \frac{t_1^{\beta-\alpha}}{\beta-\alpha} \\
&\le L \le 1.
\end{aligned}
$$

Then $G : B \to B$.

Next we show that G is a contraction on B.

Let $x_1, x_2 \in B$.

For $t \in [0, t_1]$, since $x_1, x_2 \in B$, $\| x_1 \|_\alpha, \| x_2 \|_\alpha \le \rho$. We have

$$
\begin{aligned}
&\|(Gx_1)(t) - (Gx_2)(t)\|_\alpha \\
&\le \int_0^t \| T(t-s)(f(x_1(s)) - f(x_2(s))) \|_\alpha \, ds \\
&\quad + \int_0^t \| T(t-s)[\int_{-r}^s h(s-\theta)(g(\tilde{x}_1(\theta)) - g(\tilde{x}_2(\theta)))d\theta] \|_\alpha \, ds \\
&\le K_\alpha \int_0^t (t-s)^{-\alpha} \| f(x_1(s)) - f(x_2(s)) \|_X \, ds \\
&\quad + K_\alpha \int_0^t (t-s)^{-\alpha} [\int_0^s \| h(s-\theta) \|_{L(X)} \| g(x_1(\theta)) - g(x_2(\theta)) \|_X \, d\theta] ds \\
&\le K_\alpha K_1(\rho) (\int_0^t (t-s)^{-\alpha} ds) \sup_{s \in [0,T]} \|x_1(s) - x_2(s)\|_\alpha \\
&\quad + K_\alpha K_2(\rho) (\int_{-r}^s \| h(s-\theta) \|_{L(X)} \, d\theta) (\int_0^t (t-s)^{-\alpha} ds) \sup_{s \in [0,T]} \| x_1(s) - x_2(s) \|_\alpha \\
&\le K_\alpha [K_1(\rho) + (K_2(\rho) \int_0^{T+r} \| h(\theta) \|_{L(X)} \, d\theta)] \frac{t^{1-\alpha}}{1-\alpha} \|x_1 - x_2\|_B \\
&\le K_\alpha (K_1(\rho) + K_2(\rho) \overline{h}) \frac{t_1^{1-\alpha}}{1-\alpha} \|x_1 - x_2\|_B \\
&\le L\|x_1 - x_2\|_B.
\end{aligned}
$$

Hence G is a contraction on B. By the Contraction mapping theorem, G has a unique fixed point $x \in B$, that is

$$
x(t) = (Gx)(t) = T(t)\varphi(0) + \int_0^t T(t-s) f(x(s)) ds + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta)) d\theta] ds, \quad t \in [0, t_1].
$$

Therefore x is the unique mild solution of the system (3.1.1) on $[-r, t_1]$. $\square$

Remark 3.1.5. By using strong continuity of the semigroup, we can prove the local existence of mild solutions for the system (3.1.1) without assuming $\varphi(0) \in X_\beta$, i. e., one can use

$\| T(t)\varphi(0) - \varphi(0) \|_{\alpha} = \| T(t)A^{\alpha}\varphi(0) - A^{\alpha}\varphi(0) \|_{X} \to 0,$ as $t \to 0$.

Lemma 3.1.6. Assume (A), (F1), (G1) and (H1) hold. Suppose $0 < \alpha < \beta \le 1$ and $\varphi(0) \in X_{\beta}$.

If there exists a constant $\rho > 0$ such that if $x(\bullet)$ is a possible mild solution of the system (3.1.1) on a subset $[0, T']$ of $[0, T]$ and satisfies the estimate

$$\| x(t) \|_{\alpha} \le \rho ,$$

for all $t \in [0, T']$, then there exists a constant $\rho^{*} > 0$ such that

$$\| x(t) \|_{\beta} \le \rho^{*} ,$$

for all $t \in [0, T']$.

Proof. If $x(\bullet)$ is a mild solution of the system (3.1.1) on a subset $[0, T']$ of $[0, T]$ and $\| x(t) \|_{\alpha} \le \rho$, for all $t \in [0, T']$. Then, as in the proof of Lemma 3.1.3, there exists constants $\overline{M}, \overline{N} > 0$ depending on ρ such that

$$\sup_{s \in [0,T']} \| f(x(s)) \|_{X} \le \overline{M},$$

$$\sup_{s \in [-r,T']} \| g(\tilde{x}(s)) \|_{X} \le \overline{N} .$$

Then

$$\begin{aligned}
\|x(t)\|_{\beta} &\le \| T(t)\varphi(0) \|_{\beta} + \int_{0}^{t} \| T(t-s)f(x(s)) \|_{\beta}\, ds + \int_{0}^{t} \| T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta] \|_{\beta}\, ds \\
&\le M\| \varphi(0) \|_{\beta} + K_{\beta} \int_{0}^{t} (t-s)^{-\beta} \| f(x(s)) \|_{X}\, ds \\
&\qquad + K_{\beta} \int_{0}^{t} (t-s)^{-\beta} [\int_{-r}^{s} \| h(s-\theta) \|_{L(X)} \| g(\tilde{x}(\theta)) \|_{X}\, d\theta] ds \\
&\le M \| \varphi(0) \|_{\beta} + K_{\beta} \overline{M} \frac{T^{1-\beta}}{1-\beta} + K_{\beta} \overline{N} h \frac{T'^{1-\beta}}{1-\beta} \equiv \rho^{*},
\end{aligned}$$

for all $t \in [0, T']$. The proof is complete. ☐

Theorem 3.1.7. (Extension Theorem) Assume (A), (F1), (G1) and (H1) hold.

Let $\varphi \in C([-r,0]; X_{\alpha})$ and $\varphi(0) \in X_{\beta}$, for some $\beta \in (\alpha, 1]$.

Suppose the following a priori estimate holds for the system (3.1.1):

(AP) There exists a constant $\rho > 0$ such that if $x(\cdot)$ is a possible mild solution of the system (3.1.1) on a subset $[-r, T']$ of $[-r, T]$, then $\|x(t)\|_{\alpha} \le \rho$, for all $t \in [-r, T']$.

Then the system (3.1.1) has a unique global mild solution on $[-r, T]$.

Proof. By using Lemma 3.1.6, there exists a constant ρ^{*} such that $\|x(t)\|_{\beta} \le \rho^{*}$, for all $t \in [0, T']$, whenever x is a mild solution, by the a priori estimate.

By Theorem 3.1.4, a local mild solution x_1 of the system (3.1.1) exists on $[0, t_1]$.

Then $\|x_1(t)\|_\beta \le \rho^*$, for all $t \in [-r, t_1]$. Set $\rho_1 = 1 + \rho^*$.

We must show that x_1 can be extended to be mild solution of the system (3.1.1) on [-r, T].

Given $\delta > 0$, set $B_{x_1} = \{y \in C([t_1, T]; X_\alpha) \mid y(t_1) = x_1(t_1), \| y(t) - x_1(t_1) \|_\alpha \le 1, t \in [t_1, t_1 + \delta]\}$.

Then B_{x_1} is a nonempty closed convex subset of $C([t_1, T]; X_\alpha)$.

Define a mapping G on B_{x_1} as follows: For any $y \in B_{x_1}$, define $\tilde{y}(t) = \begin{cases} y(t), t \in [t_1, t_1 + \delta], \\ x_1(t), t \in [-r, t_1], \end{cases}$

and let

$$(Gy)(t) = T(t - t_1)x_1(t_1) + \int_{t_1}^{t} T(t - t_1 - s) f(y(s))ds$$
$$+ \int_{t_1}^{t} T(t - t_1 - s)[\int_{-r}^{s} h(s - \theta) g(\tilde{y}(\theta))d\theta]ds, \; t \in [t_1, t_1 + \delta]. \qquad (3.1.7)$$

By the same argument as in Theorem 3.1.4, there exists a constant $\delta > 0$ such that

$$\begin{cases} \frac{d}{dt} y(t) + Ay = f(y(t)) + \int_{-r}^{t} h(t - s) g(y(s))ds, t \in [t_1, t_1 + \delta], \\ \qquad y(t) = x_1(t), t \in [-r, t_1], \end{cases} \qquad (3.1.8)$$

has a unique mild solution x_2 on $[t_1, t_1 + \delta]$, provided $\delta = \min\{1, T, \left(\frac{1}{L} (C_{\beta-\alpha} \rho_1 + \frac{K_\alpha \overline{K}}{\beta - \alpha}) \right)^{\frac{-1}{\beta-\alpha}} \}$

where $L \in (0, 1)$ is fixed and $\overline{K} = \overline{M} + K_1(\rho_1) + (\overline{N} + K_2(\rho_1))\overline{h}$. It is obvious that δ is only dependent on ρ_1, i. e., δ depends only on ρ.

Let $z(t) = \begin{cases} x_1(t) \text{ if } t \in [-r, t_1], \\ x_2(t) \text{ if } t \in [t_1, t_1 + \delta]. \end{cases}$

Must show that z is the unique mild solution of the system (3.1.1) on $[-r, t_1 + \delta]$.

Let w be any mild solution of the system (3.1.1) on $[-r, t_1 + \delta]$. We show that w = z on $[-r, t_1 + \delta]$.

On [-r, 0], it is obvious that w = z.

For $t \in [0, t_1]$, since x_1 is the unique mild solution on $[0, t_1]$ then $w(t) = x_1(t)$. By definition of z, $z(t) = x_1(t)$ on $[0, t_1]$. Hence $w \equiv z$ on $[0, t_1]$.

For $t \in [t_1, t_1 + \delta]$, since x_2 is the unique mild solution on $[t_1, t_1 + \delta]$ then $w(t) = x_2(t)$.

By definition of z, $z(t) = x_2(t)$ on $[t_1, t_1 + \delta]$. Hence $w \equiv z$ on $[t_1, t_1 + \delta]$.

Then z is the unique mild solution of the system (3.1.1) on $[-r, t_1 + \delta]$.

By a repeated process, since δ depends only on ρ we can extend z to $[t_1 + \delta, t_1 + 2\delta]$. By the same argument, we can obtain intervals for existence of mild solutions with equal length δ,

$[t_1, t_1+\delta]$, $[t_1+\delta, t_1+2\delta]$, ...,$[t_1+n\delta, t_1+(n+1)\delta]$ so that $T\in[t_1+n\delta, t_1+(n+1)\delta]$, for some n. Hence the system (3.1.1) has a unique global mild solution on $[-r, T]$. $\square$

We can use main idea of Theorem 3.1.4 to explain local existence of mild solutions for the following system that is more complicated than the system (3.1.1).

Consider the semilinear evolution system

$$\begin{cases} \dfrac{dx}{dt} + Ax(t) = f(t, x(t)) + \int\limits_{-r}^{t} h(t-s)g(s, x(s))ds, t \in [0, T], \\ \qquad x(t) = \varphi(t), t \in [-r, 0]. \end{cases} \tag{3.1.9}$$

Similarly, we can define classical and mild solutions to the system (3.1.9). Theorem 3.1.4 and Theorem 3.1.7 are easily extended to the following.

Theorem 3.1.8. Assume that (A), (F3), (G3), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.1.9) has a unique local mild solution.

Proof. We define a mapping G on C_φ by

$$(Gx)(t) = T(t)\varphi(0) + \int\limits_{0}^{t} T(t-s)f(s, x(s))ds + \int\limits_{0}^{t} T(t-s)[\int\limits_{-r}^{s} h(s-\theta)g(\theta, \tilde{x}(\theta))d\theta]ds, t \in [0, T],$$

where $x \in C_\varphi$ and $\tilde{x}$ is defined as in Lemma 3.1.3. (3.1.10)

Must show that $G : C_\varphi \to C_\varphi$.

Let $x \in C_\varphi$. we show that $(Gx)(t) \in X_\alpha$ for all $t \in [0, T]$.

By (F3) and continuity of x on [0, T], $f(\bullet, x(\bullet))$ is measurable on [0, T]. Since x is continuous on [0, T], $\{(s, x(s)) \mid s \in [0, T]\}$ is a bounded set in $[0, T] \times X_\alpha$. Since f maps a bounded set in $[0, T] \times X_\alpha$ to a bounded set in X, there exists a constant $\overline{M} > 0$ such that $\sup\limits_{s\in[0,T]} \|f(s, x(s))\|_X \le \overline{M}$.

Hence $f(\bullet, x(\bullet))$ is measurable and bounded on [0, T], therefore it is integrable on [0, T]. Since $f(\bullet, x(\bullet))$ is integrable, $f(s, x(s)) \in X$ and $T(t) : X \to X_\alpha$ then $\int\limits_{0}^{t} T(t-s)f(s, x(s))ds \in X_\alpha$.

By a similar argument, $g(\bullet, \tilde{x}(\bullet))$ is also measurable and bounded on [-r, T]. So it is integrable on [-r, T]. Since $h \in L_1([0, T+r]; L(X))$ then $\int\limits_{-r}^{s} h(s-\theta)g(\theta, \tilde{x}(\theta))d\theta \in X$ for all $s \in [0, T]$.

Since $T(t) : X \to X_\alpha$, then $\int\limits_{0}^{t} T(t-s)[\int\limits_{-r}^{s} h(s-\theta)g(\theta, \tilde{x}(\theta))d\theta]ds \in X_\alpha$, for all $t \in [0, T]$. This shows that each term on the right side of (3.1.10) is in X_α.

Thus $(Gx)(t) \in X_\alpha$, for all $t \in [0, T]$. Clearly, $(Gx)(0) = \varphi(0)$.

Arguing as in Lemma 3.1.3, one sees that Gx is continuous on [0, T]. Hence $Gx \in C_\varphi$. Therefore G $: C_\varphi \to C_\varphi$. Arguing as in Theorem 3.1.4, one shows that there exists $t_1 \in (0, T]$ and a closed subset B of C_φ such that G : B→B is a contraction. By the Contraction mapping theorem, the system (3.1.9) has a unique mild solution $x \in B$. □

Theorem 3.1.9. Assume that (A), (F2), (G2), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then there exists a $t_1 = t_1(\varphi) > 0$ such that the mild solution of the system (3.1.9) exists and unique on $[-r, t_1]$.

Proof. Define a mapping G as in (3.1.10). A similar process as in Theorem 3.1.4 yields a unique local mild solution x on $[-r, t_1]$ for some $t_1 = t_1(\varphi) > 0$. □

Theorem 3.1.10. Assume that (A), (F3), (G3), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Suppose a priori estimate holds for the system (3.1.9), i. e., there exists a constant $\rho > 0$ such that if $x(\cdot)$ is a possible mild solution of the system (3.1.9) on a subset $[-r, T']$ of [-r, T], the estimate $\|x(t)\|_\alpha \le \rho$ holds for all $t \in [-r, T']$, then the system (3.1.9) has a unique global mild solution on [-r, T].

Proof. By Theorem 3.1.8, the system (3.1.9) has a local mild solution x. Apply a priori estimate and a similar process as in Theorem 3.1.4 and the extension theorem, the system (3.1.9) has a unique global mild solution on [-r, T]. □

Theorem 3.1.11. Assume that (A), (F2), (G2), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Suppose a priori estimate holds for the system (3.1.9), i. e., there exists a constant $\rho^* > 0$ such that if $x(\cdot)$ is a possible mild solution of the system (3.1.9) on a subset $[-r, T']$ of [-r, T], the estimate $\|x(t)\|_\alpha \le \rho^*$ holds for all $t \in [-r, T']$, then the system (3.1.9) has a unique global mild solution on [-r, T].

Proof. By Theorem 3.1.9, the system (3.1.9) has a local mild solution x. Apply a priori estimate and a similar process as in Theorem 3.1.4 and the extension theorem, the system (3.1.9) has a unique global mild solution on [-r, T]. □

3.2 A Priori Estimate and Global Existence of Mild Solutions

Lemma 3.2.1. (Gronwall's Lemma with Singularity and Time Lag)

Let $C = C([0, T']; X_\alpha)$ and $x \in C$ satisfies the following inequality

$$\|x(t)\|_\alpha \le a + b\int_0^t (t-s)^{-\alpha} \|x(s)\|_\alpha \, ds + c\int_0^t (t-s)^{-\alpha} \|x_s\|_C \, ds, \; t \in [0, T'], \qquad (3.2.1)$$

where $a, b, c \ge 0$ are constants and $\|x_s\|_C = \sup_{0\le\xi\le s} \|x(\xi)\|_\alpha$. Then there exits a constant $M_1 > 0$ (independent of a) such that

$$\|x(t)\|_\alpha \le M_1 a,$$

for all $t \in [0, T']$.

Proof. Define $v(t) = \int_0^t (t-s)^{-\alpha} \|x_s\|_C \, ds = \int_0^t \theta^{-\alpha} \|x_{t-\theta}\|_C \, d\theta$.

We show that $v(\cdot)$ is monotonously increasing on $[0, T']$.

Let $0 \le t_1 \le t_2 \le T'$. Then

$$\begin{aligned} v(t_1) - v(t_2) &= \int_0^{t_1} \theta^{-\alpha} \|x_{t_1-\theta}\|_C \, d\theta - \int_0^{t_2} \theta^{-\alpha} \|x_{t_2-\theta}\|_C \, d\theta \\ &= \int_0^{t_1} \theta^{-\alpha} (\|x_{t_1-\theta}\|_C - \|x_{t_2-\theta}\|_C) d\theta - \int_{t_1}^{t_2} \theta^{-\alpha} \|x_{t_2-\theta}\|_C \, d\theta. \end{aligned}$$

Since $t_1 - \theta \le t_2 - \theta$, $v(t_1) - v(t_2) \le 0$, hence v is monotonously increasing on $[0, T']$.

Since v is increasing on $[0, T']$ and $\|x(s)\|_\alpha \le \|x_s\|_C$, we have

$$\begin{aligned} \|x_t\|_C &= \sup_{0\le\xi\le t} \|x(\xi)\|_\alpha \\ &\le \sup_{0\le\xi\le t} [a + b\int_0^\xi (\xi-s)^{-\alpha} \|x(s)\|_\alpha \, ds + c\int_0^\xi (\xi-s)^{-\alpha} \|x_s\|_C \, ds] \\ &\le \sup_{0\le\xi\le t} [a + c_1\int_0^\xi (\xi-s)^{-\alpha} \|x_s\|_C \, ds] \\ &\le \sup_{0\le\xi\le t} [a + c_1 v(\xi)] \le a + c_1 v(t). \end{aligned}$$

So $\|x_t\|_C \le a + c_1 \int_0^t (t-s)^{-\alpha} \|x_s\|_C \, ds$. By Gronwall's lemma (Corollary 2.5.3), there exists a constant $M_1 > 0$ (independent of a) such that $\|x_t\|_C \le M_1 a$, for all $t \in [0, T']$.

Since $\|x(t)\|_\alpha \le \|x_t\|_C$ then $\|x(t)\|_\alpha \le M_1 a$, for all $t \in [0, T']$. Then the proof is complete. □

By virtue of the Gronwall's lemma with singularity and time lag, together with linear growth condition, we can prove the following global existence theorem without assuming a priori estimate.

Theorem 3.2.2. (Global Existence Theorem) Assume that (A), (F1), (F5), (G1), (G5) and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.1.1) has a unique global mild solution on [-r, T].

Proof. We show a priori estimate holds, i. e., there exists a constant $\rho > 0$ such that if $x(\cdot)$ is a mild solution of the system (3.1.1) on a subset $[-r, T']$, $T' \in [0, T]$, it follows that

$$\| x(t) \|_\alpha \le \rho ,$$

for all $t \in [-r, T']$.

Suppose $x(\cdot)$ is a mild solution of the system (3.1.1) on a subset $[-r, T']$ of $[-r, T]$.

For $t \in [0, T']$, since $x(t)$ is a mild solution of the system (3.1.1) and satisfies the equation (3.1.2) on $[0, T']$, by using assumption (F5) and (G5), it follows that

$$\begin{aligned}
\|x(t)\|_\alpha \le\ & \|T(t)\varphi(0)\|_\alpha + \int_0^t \| T(t-s) f(x(s)) \|_\alpha \, ds \\
& + \int_0^t \| T(t-s)[\int_{-r}^s h(s-\xi) g(x(\xi)) d\xi] \|_\alpha \, ds \\
\le\ & M\|\varphi(0)\|_\alpha + K_\alpha \int_0^t (t-s)^{-\alpha} \, \| f(x(s)) \|_X \, ds \\
& + K_\alpha \int_0^t (t-s)^{-\alpha} [\int_{-r}^s \| h(s-\xi) g(x(\xi)) \|_X \, d\xi] ds \\
\le\ & M\|\varphi(0)\|_\alpha + K_\alpha \int_0^t (t-s)^{-\alpha} (K_1 (1 + \| x(s) \|_\alpha)) ds \\
& + K_\alpha \int_0^t (t-s)^{-\alpha} [\int_{-r}^s \| h(s-\xi) \|_{L(X)} \, (K_2 (1 + \| x(\xi) \|_\alpha)) d\xi] ds \\
\le\ & M\|\varphi(0)\|_\alpha + K_\alpha K_1 \int_0^t (t-s)^{-\alpha} ds \\
& + K_\alpha K_1 \int_0^t (t-s)^{-\alpha} \, \| x(s) \|_\alpha \, ds \\
& + K_\alpha K_2 (\int_0^t (t-s)^{-\alpha} ds)(\int_{-r}^T \| h(T-\xi) \|_{L(X)} \, d\xi) \\
& + K_\alpha K_2 \int_0^t (t-s)^{-\alpha} [\int_{-r}^s \| h(s-\xi) \|_{L(X)} \| x(\xi) \|_\alpha \, d\xi] ds \\
\le\ & M\|\varphi(0)\|_\alpha + K_\alpha K_1 \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_2 \, \bar{h} \, \frac{T'^{1-\alpha}}{1-\alpha} \\
& + K_\alpha K_1 \int_0^t (t-s)^{-\alpha} \, \| x(s) \|_\alpha \, ds \\
& + K_\alpha K_2 \int_0^t (t-s)^{-\alpha} [\int_{-r}^0 \| h(s-\xi) \|_{L(X)} \| \varphi(\xi) \|_\alpha \, d\xi + \int_0^s \| h(s-\xi) \|_{L(X)} \| x(\xi) \|_\alpha \, d\xi] ds \\
\le\ & M \| \varphi(0) \|_\alpha + K_\alpha K_1 \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_2 \, \bar{h} \, \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_1 \int_0^t (t-s)^{-\alpha} \, \| x(s) \|_\alpha \, ds \\
& + K_\alpha K_2 \int_0^t [\int_{-r}^0 \| h(s-\xi) \|_{L(X)} \| \varphi(\xi) \|_\alpha \, d\xi] (t-s)^{-\alpha} ds \\
& + K_\alpha K_2 \int_0^t (t-s)^{-\alpha} \int_0^s \| h(s-\xi) \|_{L(X)} \, d\xi \sup_{0 \le \xi \le s} \| x(\xi) \|_\alpha \, ds
\end{aligned}$$

$$\leq M\|\varphi(0)\|_\alpha + K_\alpha K_1 \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_2 \bar{h}\, \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_1 \int_0^t (t-s)^{-\alpha} \|x(s)\|_\alpha \, ds$$

$$+ K_\alpha K_2 \bar{h}\, \|\varphi\|_{C([-r,0];X_\alpha)} \frac{T'^{1-\alpha}}{1-\alpha}$$

$$+ K_\alpha K_2 \bar{h} \int_0^t (t-s)^{-\alpha} \sup_{0\leq\xi\leq s} \|x(\xi)\|_\alpha \, ds$$

$$\leq a + b\int_0^t (t-s)^{-\alpha} \|x(s)\|_\alpha \, ds + c\int_0^t (t-s)^{-\alpha} \|x_s\|_C \, ds,$$

where $a = M\|\varphi(0)\|_\alpha + K_\alpha K_1 \frac{T'^{1-\alpha}}{1-\alpha} + K_\alpha K_2 \bar{h}\, \|\varphi\|_{C([-r,0];X_\alpha)} \frac{T'^{1-\alpha}}{1-\alpha}$, $b = K_\alpha K_1$, $c = K_\alpha K_2 \bar{h}$, and $C = C([0,T'];X_\alpha)$. By Gronwall's lemma with singularity and time lag, there exists a constant $M_1 > 0$ (independent of a) such that

$$\|x(t)\|_\alpha \leq M_1 a,$$

for all $t \in [0, T']$.

On [-r, 0], $\|x(t)\|_\alpha = \|\varphi(t)\|_\alpha \leq \|\varphi\|_{C([-r,0];X_\alpha)}$. Let $\rho = \max\{M_1 a, \|\varphi\|_{C([-r,0];X_\alpha)}\}$.

Then

$$\|x(t)\|_\alpha \leq \rho,$$

for all $t \in [-r, T']$.

By Theorem 3.1.4, the system (3.1.1) has a local mild solution x, combining the extension theorem and the a priori estimate, the mild solution x can be extended to [-r, T]. □

We consider another type of global existence problem. Now we will deal with super linear growth conditions. The following theorem shows that an a priori estimate for the α- norm of solution can be obtained, provided the function f and g satisfy a super linear growth condition and we know an a priori estimate in some weaker norm.

Theorem 3.2.3. Assume that (A), (F1), (F6), (G1), (G6) and (H1) hold.

Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\lambda\alpha, 1]$, suppose, there exists a constant $\rho > 0$ such that if $x(\bullet)$ is a possible mild solution of the system (3.1.1) on a subset $[-r, T']$ of [-r, T], then

$$\|x(t)\|_E \leq \rho,$$

for all $t \in [-r, T']$. Then there exists a constant $\rho^* > 0$ such that

$$\|x(t)\|_\alpha \leq \rho^*,$$

for all $t \in [-r, T']$, hence the system (3.1.1) has a unique global mild solution on [-r, T].

Proof. Since $\lambda \in [1, \frac{1}{\alpha})$ then $\alpha \le \lambda\alpha < \beta \le 1$. Let $\gamma = \lambda\alpha$. The embedding relation

$$X_\beta \ ▪\ X_\gamma \ ▪\ X_\alpha \ ▪\ E \ ▪\ X,$$

is true.

Let $\rho > 0$. Suppose $x(\bullet)$ is a mild solution of the system (3.1.1) on $[-r, T']$ with $\| x(t) \|_E \le \rho$, $t \in [-r, T']$. This means

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s) f(x(s))ds + \int_0^t T(t-s)[\int_{-r}^s h(t-s) g(x(\theta))d\theta]ds, t \in [0, T'], \\ \varphi(t), t \in [-r, 0], \end{cases}$$

and

$$\| x(t) \|_E \le \rho,$$

for all $t \in [-r, T']$.

By the " moment inequality", there exists a constant $M_{\alpha,\gamma}$ such that

$$\|x(s)\|_\alpha \le M_{\alpha,\gamma}(\|x(s)\|_\gamma)^{1/\lambda} \|x(s)\|_X^{\frac{\lambda-1}{\lambda}},$$

for $s \in [-r, T']$.

In addition, since E O X and X_β O X_γ, it follows that

$$\begin{aligned} \|x(s)\|_\alpha^\lambda &\le M_{\alpha,\gamma}^\lambda \|x(s)\|_\gamma \|x(s)\|_X^{\lambda-1} \\ &\le N_{\alpha,\beta}^\lambda \|x(s)\|_\beta \|x(s)\|_E^{\lambda-1}, \end{aligned}$$

for $s \in [-r, T']$.

Let $t \in [0, T']$. Then

$$\begin{aligned} \|x(t)\|_\beta &\le \|T(t)\varphi(0)\|_\beta + \int_0^t \| T(t-s) f(x(s)) \|_\beta \, ds \\ &\quad + \int_0^t \| T(t-s)[\int_{-r}^s h(s-\theta) g(\tilde{x}(\theta))d\theta] \|_\beta \, ds \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where $\tilde{x}$ is defined as in Lemma 3.1.3.

Since $\varphi(0) \in X_\beta$ then

$$I_1 \le \|A^\beta T(t)\varphi(0)\|_X = \|T(t)A^\beta \varphi(0)\|_X \le M\|\varphi(0)\|_\beta.$$

Since $\| x(t) \|_E \le \rho$ for all $t \in [0, T']$, by (F6) and Theorem 2.3.16 (c), there exists constants K_β and

$c(\rho) > 0$ such that

$$\begin{aligned} I_2 &\le K_\beta \int_0^t (t-s)^{-\beta} [c(\rho)(1 + \|x(s)\|_\alpha^\lambda)]ds \\ &\le K_\beta \, c(\rho) \int_0^t (t-s)^{-\beta} ds + K_\beta \, c(\rho) \int_0^t (t-s)^{-\beta} \|x(s)\|_\alpha^\lambda \, ds \\ &\le K_\beta \, c(\rho) \frac{T'^{1-\beta}}{1-\beta} + K_\beta \, c(\rho) \int_0^t (t-s)^{-\beta} (N_{\alpha,\beta}^\lambda \|x(s)\|_\beta \| x(s) \|_E^{\lambda-1})ds \\ &\le K_\beta \, c(\rho) \frac{T'^{1-\beta}}{1-\beta} + K_\beta \, c(\rho) N_{\alpha,\beta}^\lambda \rho^{\lambda-1} \int_0^t (t-s)^{-\beta} \|x(s)\|_\beta \, ds \end{aligned}$$

Similarly, by (G6) we have a constant $d(\rho) > 0$ such that

$$
\begin{aligned}
I_3 &\le \int_0^t \| T(t-s)[\int_{-r}^0 h(s-\theta)g(\varphi(\theta))d\theta] \|_\beta \, ds + \int_0^t \| T(t-s)[\int_0^s h(s-\theta)g(x(\theta))d\theta] \|_\beta \, ds \\
&\le K_\beta \int_0^t (t-s)^{-\beta} [\int_{-r}^0 \| h(s-\theta) \|_{L(X)} \| g(\varphi(\theta)) \|_X \, d\theta] ds \\
&\quad + K_\beta \, d(\rho) \int_0^t (t-s)^{-\beta} [\int_0^s \|h(s-\theta)\|_{L(X)}(1 + \| x(\theta)\|_\alpha^\lambda) d\theta] ds \\
&\le K_\varphi + K_\beta \, d(\rho) \int_0^t (t-s)^{-\beta} [\int_0^s \|h(s-\theta)\|_{L(X)}(1 + \| x(\theta)\|_\alpha^\lambda) d\theta] ds \\
&\le K_\varphi + K_\beta \, d(\rho) \int_0^t (t-s)^{-\beta} (\int_0^s \|h(s-\theta)\|_{L(X)} d\theta) ds \\
&\quad + K_\beta \, d(\rho) \int_0^t (t-s)^{-\beta} [\int_0^s \|h(s-\theta)\|_{L(X)} \| x(\theta) \|_\alpha^\lambda \, d\theta] ds \\
&\le K_\varphi + K_\beta \, d(\rho) \, \bar{h} \int_0^t (t-s)^{-\beta} ds \\
&\quad + K_\beta \, d(\rho) N_{\alpha,\beta}^\lambda \int_0^t (t-s)^{-\beta} [\int_0^s \|h(\theta)\|_{L(X)} (\| x(\theta) \|_E^{\lambda-1} \| x(\theta) \|_\beta) \, d\theta] ds \\
&\le K_\varphi + K_{h,\rho} + K_\beta \, d(\rho) N_{\alpha,\beta}^\lambda \, \bar{h} \, \rho^{\lambda-1} \int_0^t (t-s)^{-\beta} \sup_{0 \le \theta \le s} \|x(\theta)\|_\beta \, ds \\
&\le K_\varphi + K_{h,\rho} + K_\beta \, d(\rho) N_{\alpha,\beta}^\lambda \, \bar{h} \, \rho^{\lambda-1} \int_0^t (t-s)^{-\beta} \|x_s\|_C ds,
\end{aligned}
$$

where $K_\varphi > 0$ is a constant depending only on φ, $K_{h,\rho} = K_\beta \, d(\rho) \, \bar{h} \dfrac{T'^{1-\beta}}{1-\beta}$, and $C = C([0, T']; X_\alpha)$.

Then

$$\|x(t))\|_\beta \le a + b \int_0^t (t-s)^{-\beta} \|x(s)\|_\beta \, ds + c \int_0^t (t-s)^{-\beta} \|x_s\|_C ds,$$

where $a = M\|\varphi(0)\|_\beta + K_\beta \, c(\rho) \dfrac{T'^{1-\beta}}{1-\beta} + K_\varphi + K_{h,\rho}$, $b = K_\beta \, c(\rho) N_{\alpha,\beta}^\lambda \rho^{\lambda-1}$, $c = K_\beta \, d(\rho) N_{\alpha,\beta}^\lambda \bar{h} \rho^{\lambda-1}$.

By the Gronwall's Lemma with singularity and time lag (Lemma 3.2.1), there exists a constant $M_1 > 0$ such that

$$\|x(t)\|_\beta \le M_1 a.$$

Then $\|x(t)\|_\alpha \le c_1 M_1 a$, for all $t \in [0, T']$. Set $\rho^* = \max\{c_1 M_1 a, \| \varphi \|_{C([-r,0];X_\alpha)}\}$.

Thus $\| x(t) \|_\alpha \le \rho^*$, for all $t \in [-r, T']$.

By assumptions and Theorem 3.1.4, the system (3.1.1) has a unique local mild solution x. Combining the extension theorem and the a priori estimate x can be extended to [-r, T]. □

3.3 Regularity of Mild Solutions

In the following we discuss the regularity of mild solutions. We study the connection between mild solution and classical solution. It can be seen that under some stronger assumptions, the mild solution is a classical one.

Theorem 3.3.1. (Regularity) Assume that (A), (F4), (G4), and (H2) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. If a mild solution x of the system (3.1.9) exists on [-r, T], then
$x \in C([-r, T]; X_\alpha) \cap C^1((0, T); X)$, hence it is a classical solution.

Proof. Suppose the system (3.1.9) has a mild solution $x \in C([-r, T]; X_\alpha)$. Then

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta)g(\theta, x(\theta))d\theta]ds, t \in [0, T], \\ \varphi(t), t \in [-r, 0]. \end{cases}$$

(3.3.1)

Define

$$y(t) = \begin{cases} T(t)A^{\alpha}\varphi(0) + \int_0^t A^{\alpha}T(t-s)f(s, x(s))ds + \int_0^t A^{\alpha}T(t-s)[\int_{-r}^s h(s-\theta)g(\theta, x(\theta))d\theta]ds, t \in [0, T], \\ A^{\alpha}\varphi(t), t \in [-r, 0]. \end{cases} \quad (3.3.2)$$

It is easy to see that $y \in C([-r, T]; X)$

We prove that y is locally Hölder continuous on (0, T].

Firstly, we show that t # $f(t, A^{-\alpha}y(t))$ is continuous on [0, T].

Since f is locally Hölder continuous in $t \in [0, T]$, locally Lipschitz in $x \in X_\alpha$ and $y \in C([-r, T]; X)$ then for each $t \in [0, T]$, for a fixed $\rho > 0$ there exists constants $\nu \in (0,1)$, $L > 0$ and $K_1 = K_1(t, \rho) > 0$ such that

$$\begin{aligned} &\|f(t, A^{-\alpha}y(t)) - f(s, A^{-\alpha}y(s))\|_X \\ &\quad \le \|f(t, A^{-\alpha}y(t)) - f(s, A^{-\alpha}y(t))\|_X + \|f(s, A^{-\alpha}y(t)) - f(s, A^{-\alpha}y(s))\|_X \\ &\quad \le L\|t-s\|^{\nu} + K_1 \|A^{-\alpha}y(t) - A^{-\alpha}y(s)\|_\alpha \\ &\quad \le L\|t-s\|^{\nu} + K_1 \|y(t) - y(s)\|_X \end{aligned}$$

Then t # $f(t, A^{-\alpha}y(t))$ is continuous on [0, T]. Therefore it is bounded on [0, T].

Then there exists a constant N_1 such that $\|f(t, A^{-\alpha}y(t))\|_X \le N_1$, for $t \in [0, T]$.

By the same argument as f, t # $g(t, A^{-\alpha}y(t))$ is continuous on [–r, T].

Then there exists a constant N_2 such that $\|g(t, A^{-\alpha}y(t))\|_X \le N_2$, for $t \in [-r, T]$.

By the continuity of t # $g(t, A^{-\alpha}y(t))$ on [–r, T] and $h \in L_p([0, T + r]; L(X))$, we have $\int_{-r}^t h(t-s)g(s, A^{-\alpha}y(s))ds \in L_1([0, T + r]; X)$.

Thirdly, let $t \in (0, T)$. Choose $0 < \delta < 1$ such that $(t - \frac{\delta}{2}, t + \frac{\delta}{2}) \subset (0, T]$.

Let $s_1, s_2 \in (t - \frac{\delta}{2}, t + \frac{\delta}{2})$. Suppose $s_1 < s_2$ and let $\xi = s_2 - s_1$. Then $0 < \xi < 1$ and

$$\begin{aligned} &\|y(s_1 + \xi) - y(s_1)\|_X \\ &\qquad \le \|T(s_1 + \xi)A^{\alpha}\varphi(0) - T(s_1)A^{\alpha}\varphi(0)\|_X \\ &\qquad\quad + \|\int_0^{s_1+\xi} T(s_1 + \xi - \theta)A^{\alpha}f(\theta, A^{-\alpha}y(\theta))d\theta - \int_0^{s_1} T(s_1 - \theta)A^{\alpha}f(\theta, A^{-\alpha}y(\theta))d\theta\|_X \\ &\qquad\quad + \|\int_0^{s_1+\xi} T(s_1 + \xi - \theta)A^{\alpha}[\int_{-r}^{\theta} h(\theta - \tau)g(\tau, A^{-\alpha}y(\tau))d\tau]d\theta \end{aligned}$$

$$
\begin{aligned}
& - \int_0^{s_1} T(s_1-\theta)A^{\alpha}[\int_{-r}^{\theta} h(\theta-\tau)g(\tau, A^{-\alpha}y(\tau))d\tau]d\theta \,\|_X \\
\leq\ & \|(T(\xi)-I)T(s_1)A^{\alpha}\varphi(0)\|_X \\
& + \int_0^{s_1} \| (T(\xi)-I)A^{\alpha}T(s_1-\theta)f(\theta, A^{-\alpha}y(\theta)) \|_X \, d\theta \\
& + \int_{s_1}^{s_1+\xi} \| A^{\alpha}T(s_1+\xi-\theta)f(\theta, A^{-\alpha}y(\theta)) \|_X \, d\theta \\
& + \int_0^{s_1} \| (T(\xi)-I)A^{\alpha}T(s_1-\theta)[\int_{-r}^{\theta} h(\theta-\tau)g(\tau, A^{-\alpha}y(\tau))d\tau] \|_X \, d\theta \\
& + \int_{s_1}^{s_1+\xi} \| A^{\alpha}T(s_1-\theta)[\int_{-r}^{\theta} h(\theta-\tau)g(\tau, A^{-\alpha}y(\tau))d\tau] \|_X \, d\theta \\
=\ & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Choose $\gamma \in (0, 1-\alpha)$, by Theorem 2.3.16 (c), (d), we have

$$
\begin{aligned}
I_1 \quad & \leq \quad \|(T(\xi)-I)T(s_1)\, A^{\alpha}\varphi(0)\|_X \\
& \leq \quad C_\gamma\, \xi^\gamma \,\| A^\gamma\, T(s_1)A^{\alpha}\varphi(0)\|_X, \\
& \leq \quad C_\gamma\, \xi^\gamma\, K_\gamma\, s_1^{-\gamma}\, \|A^{\alpha}\varphi(0)\|_X \\
& \leq \quad C_\gamma\, K_\gamma\, s_1^{-\gamma}\, \|\varphi(0)\|_\alpha\, \xi^\gamma \;\equiv\; M_1\xi^\gamma.
\end{aligned}
$$

By a similar argument, we have

$$
\begin{aligned}
I_2 \quad & \leq \quad C_\gamma\, \xi^\gamma \int_0^{s_1} \| A^{\alpha+\gamma}T(s_1-\theta)f(\theta, A^{-\alpha}y(\theta)) \|_X \, d\theta \\
& \leq \quad C_\gamma\, N_1 K_{\alpha+\gamma}\, \xi^\gamma \int_0^{s_1} (s_1-\theta)^{-(\alpha+\gamma)} d\theta \\
& \leq \quad C_\gamma\, N_1 K_{\alpha+\gamma}\, \xi^\gamma \, \frac{T^{1-(\alpha+\gamma)}}{1-(\alpha+\gamma)} \\
& \equiv \quad M_2\xi^\gamma.
\end{aligned}
$$

We have

$$
I_3 \quad \leq \quad K_\alpha N_1 \int_{s_1}^{s_1+\xi} (s_1+\xi-\theta)^{-\alpha} d\theta \;=\; \frac{K_\alpha N_1}{1-\alpha}\, \xi^{1-\alpha} \;\leq\; \frac{K_\alpha N_1}{1-\alpha}\, \xi^\gamma \equiv M_3\xi^\gamma.
$$

Similarly,

$$
\begin{aligned}
I_4 \quad & \leq \quad C_\gamma\, K_{\alpha+\gamma}\, \xi^\gamma\, N_2 \int_0^{s_1} (s_1-\theta)^{-\gamma} d\theta \int_{-r}^{T} \| h(T-s) \|_{L(X)}\, ds \\
& \leq \quad C_\gamma\, K_\alpha N_2\, \frac{T^{1-\gamma}}{1-\gamma}\, \bar{h}\, \xi^\gamma \\
& \equiv \quad M_4\xi^\gamma.
\end{aligned}
$$

Similarly,

$$
I_5 \quad \leq \quad K_\alpha N_2 \,(\int_{-r}^{T} \| h(T-s) \|_{L(X)}\, ds) \int_{s_1}^{s_1+\xi} (s_1+\xi-\theta)^{-\alpha} d\theta
$$

$$\begin{aligned} &\leq \quad K_\alpha N_2 \bar{h}\,\frac{\xi^{1-\alpha}}{1-\alpha} \\ &\leq \quad K_\alpha N_2 \bar{h}\,\frac{\xi^{\gamma}}{1-\gamma} \\ &\equiv \quad M_5\,\xi^{\gamma}. \end{aligned}$$

Then $\|y(s_1+\xi)-y(s_1)\|_X \leq (M_1+M_2+M_3+M_4+M_5)\,\xi^{\gamma} \equiv L\xi^{\gamma}$.

Hence y is locally Hölder continuous in $t\in(0, T)$. The continuity of y at the end point also holds in a similar way. Therefore y is locally Hölder continuous in $t\in(0, T]$.

Locally Hölder continuity of $t \# f(t, A^{-\alpha}y(t))$ on [0, T] can be shown easily by using the following

$$\begin{aligned} \|f(s_1, A^{-\alpha}y(s_1)) - f(s_2, A^{-\alpha}y(s_2))\|_X &\leq L_1(|s_1-s_2|^{\theta 1} + \|A^{-\alpha}y(s_1) - A^{-\alpha}y(s_2)\|_\alpha) \\ &\leq L_1(|s_1-s_2|^{\theta 1} + \|y(s_1)-y(s_2)\|_X) \\ &\leq L_1(|s_1-s_2|^{\theta 1} + L_2|s_1-s_2|^{\gamma}) \\ &\leq L_3|s_1-s_2|^{\eta}, \end{aligned}$$

$\eta = \min\{\theta_1, \gamma\}$, L_1, L_2 and L_3 are constants.

To show $t \to \int_{-r}^{t} h(t-s)g(s, A^{-\alpha}y(s))ds$ is locally Hölder continuous in $t\in(0, T]$.

Since g is locally Hölder continuous in $t\in[-r, T]$ and y is locally Hölder continuous in $t\in(0, T]$, for any $t\in(0, T]$, there is a $\delta>0$ such that g and y are Hölder continuous in $V = (t-\delta, t+\delta)\subset(0, T]$. So there are constants θ_2, $\gamma_2\in(0, 1)$ and L_4, $L_5 > 0$ such that for any ℓ_1, ℓ_2 in V, say $\ell_1<\ell_2$,

$$\begin{aligned} &\|\int_{-r}^{\ell_1} h(\ell_1-s)g(s, A^{-\alpha}y(s))ds - \int_{-r}^{\ell_2} h(\ell_2-s)g(s, A^{-\alpha}y(s))ds\|_X \\ &\quad = \|\int_0^{\ell_1+r} h(z)g(\ell_1-z, A^{-\alpha}y(\ell_1-z))dz - \int_0^{\ell_2+r} h(z)g(\ell_2-z)g(\ell_2-z, A^{-\alpha}y(\ell_2-z))dz\|_X \\ &\quad \leq \quad \|\int_0^{\ell_1+r} h(z)(g(\ell_1-z, A^{-\alpha}y(\ell_1-z)) - g(\ell_2-z, A^{-\alpha}y(\ell_2-z)))dz\|_X \\ &\qquad\qquad + \|\int_{\ell_1+r}^{\ell_2+r} h(z)g(\ell_2-z, A^{-\alpha}y(\ell_2-z))dz\|_X \\ &\quad \leq \int_0^{\ell_1+r} \|h(z)\|_{L(X)}(L_4|\ell_1-\ell_2|^{\theta 2} + L_5|\ell_1-\ell_2|^{\gamma 2})dz + N_2[\int_{\ell_1+r}^{\ell_2+r}\|h(z)\|^p dz]^{\frac{1}{p}}(\int_{\ell_1+r}^{\ell_2+r} 1^q dz) \end{aligned}$$

$\frac{1}{q}$

$$\begin{aligned} &\quad \leq (\int_0^{T+r}\|h(z)\|_{L(X)}dz)(L_4|\ell_1-\ell_2|^{\theta 2} + L_5|\ell_1-\ell_2|^{\gamma 2}) + N_2[\int_{\ell_1+r}^{\ell_2+r}\|h(z)\|^p dz]^{\frac{1}{p}}|\ell_1-\ell_2|^{\frac{1}{q}} \\ &\quad \leq (\int_0^{T+r}\|h(z)\|_{L(X)}^p dz)^{\frac{1}{p}}(\int_0^{T+r}1^q dz)^{\frac{1}{q}}(L_6|\ell_1-\ell_2|^{\eta}) + N_2K_1|\ell_1-\ell_2|^{\frac{1}{q}} \\ &\quad \leq K_2|\ell_1-\ell_2|^{\kappa}, \end{aligned}$$

where K_2 is a constant, $\eta=\min\{\theta_2, \gamma_2\}$, $\kappa = \min\{\eta, \frac{1}{q}\} = \min\{\theta_2, \gamma_2, (p-1)/p\}$, By Corollary 2.4.17, since –A generates an analytic semigroup T(t), t # f(t, $A^{-\alpha}$y(t) is locally Hölder continuous in $t \in [0, T]$ and $t \to \int_{-r}^{t} h(t-s)g(s, A^{-\alpha}y(s))ds$ is locally Hölder continuous in $t \in (0, T]$, the system

$$\begin{cases} (d/dt)w(t) + Aw(t) = f(t, A^{-\alpha}y(t)) + \int_{-r}^{t} h(t-s)g(s, A^{-\alpha}y(s))ds, \; t \in [0, T], \\ \qquad\qquad w(t) = \varphi(t), t \in [-r, 0]. \end{cases}$$

(3.3.3)

has a unique classical solution $w \in C([-r, T]; X_\alpha) \cap C^1((0,T), X)$.

Rearrange form of w, we obtain

$$\begin{aligned} w(t) &= T(t)\varphi(0) + \int_0^t T(t-s)f(s, A^{-\alpha}y(s))ds + \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\theta, A^{-\alpha}y(\theta))d\theta]ds \\ &= T(t)\varphi(0) + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\theta, \tilde{x}(\theta))d\theta]ds \\ &= x(t), \; t \in [0, T]. \end{aligned}$$

Then $x \in C^1((0, T); X)$. Hence $x \in C([-r, T]; X_\alpha) \cap C^1((0, T); X)$ is a classical solution of the system (3.1.9).

□

Corollary 3.3.2. Assume that (A1), (F4), (F5), (G4), (G5) and (H2) hold.
Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.1.9) has a unique classical solution.

Proof. Since[0,T + r] is a bounded domain then (H2) implies (H1). By assumptions, the system (3.1.9) has a local mild solution x. Applying the growth condition (F5) and (G5), by Theorem 3.2.2 x can be extended to [0, T]. By Theorem 3.3.1, the solution $x \in C^1((0, T); X_\alpha))$. Hence x is the unique classical solution of the system (3.1.9).

□

We give a remark here in order to show locally Hölder continuity of mild solutions of the system (3.1.1).

Remark 3.3.3. The mapping G in Lemma 3.1.3, maps C_φ into $C^\theta([0, T]; X_\alpha)$ for some $\theta \in (0, 1)$, provided that $\varphi(0) \in X_\beta$ for some β such that $0 < \alpha < \beta < 1$.

Proof. Let $\varphi(0) \in X_\beta$, $0 < \alpha < \beta < 1$.

Recall that $C_\varphi = \{x \in C([0,T]; X_\alpha) \mid x(0) = \varphi(0)\}$.

Let $x \in C_\varphi$. We show that $Gx \in C^\theta([0, T]; X_\alpha)$ for some $\theta \in (0, 1)$.

Let $0 \le t < t + \xi \le T$ and $0 < \xi < 1$. Then

$$\|(Gx)(t+\xi) - (Gx)(t)\|_\alpha$$

$$\begin{aligned} &\le \|T(t+\xi)\varphi(0) - T(t)\varphi(0)\|_\alpha \\ &+ \|\int_0^{t+\xi} T(t+\xi-s)f(x(s))ds - \int_0^t T(t-s)f(x(s))ds\|_\alpha \\ &+ \|\int_0^{t+\xi} T(t+\xi-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta]ds - \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\tilde{x}(\theta))d\theta]ds\|_\alpha \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Since $\varphi(0) \in X_\beta = D(A^{\beta-\alpha}A^\alpha)$, then $A^\alpha\varphi(0) \in D(A^{\beta-\alpha}) = X_{\beta-\alpha}$. By using Theorem 2.3.16 (c),(d), and the same procedures in Theorem 3.3.1, one can estimate each I_i's by a constant multiple of $\xi^{\beta-\alpha}$. This shows that Gx is Hölder continuous in [0,T] with exponent $\theta \equiv \beta - \alpha \in (0, 1)$. □

3.4 Continuous Dependence

Theorem 3.4.1. Assume that the hypotheses of Theorem 3.2.2 are satisfied. For any $\rho > 0$, if x and y are mild solutions of the system (3.1.1) on [-r, T] corresponding to φ_1 and φ_2, respectively, then there exists a constant $K(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \le K(\rho)\|\varphi_1 - \varphi_2\|_{C([-r,0];X_\alpha)},$$

provided $\varphi_1, \varphi_2 \in C([-r,0];X_\alpha)$ with $\|\varphi_1\|_{C([-r,0];X_\alpha)} \le \rho$ and $\|\varphi_2\|_{C([-r,0];X_\alpha)} \le \rho$.

Proof. First, we show that any mild solution z of the system (3.1.1) on [-r, T] corresponding to $\varphi \in C([-r, 0]; X_\alpha)$ with $\|\varphi\|_{C([-r,0];X_\alpha)} \le \rho$, satisfies the estimate

$$\|z\|_{C([-r,T];X_\alpha)} \le \rho^*,$$

where ρ^* is a constant depending only on ρ.

Proceeding as in the proof of Theorem 3.2.2, it is easy to verify that there exists a constant $\rho_1 > 0$ such that

$$\|z(t)\|_\alpha \le \rho_1,$$

for $t \in [0, T]$.

Set $\rho^* = \max\{\rho_1, \rho\}$. We have $\|z\|_{C([-r,T];X_\alpha)} \le \rho^*$.

Thus in particular, $\|x\|_{C([-r,T];X_\alpha)} \le \rho^*$, and $\|y\|_{C([-r,T];X_\alpha)} \le \rho^*$,

Next we show that there exists a constant $K(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \le K(\rho)\|\varphi_1 - \varphi_2\|_{C([-r,0];X_\alpha)}.$$

For $t \in [-r, 0]$, it is easy to see that

$$\|x(t) - y(t)\|_\alpha \le \|(\varphi_1 - \varphi_2)(t)\|_\alpha \le \|\varphi_1 - \varphi_2\|_{C([-r,0];X_\alpha)}.$$

For $t \in [0, T]$, we have

$$\begin{aligned}\|x(t) - y(t)\|_\alpha \le\ & \|T(t)(\varphi_1 - \varphi_2)(0)\|_\alpha \\ & + \int_0^t \|T(t-s)(f(x(s)) - f(y(s)))\|_\alpha ds \\ & + \int_0^t \|T(t-s)[\int_{-r}^s h(s-\theta)(g(x(\theta)) - g(y(\theta)))d\theta]\|_\alpha ds \\ =\ & I_1 + I_2 + I_3.\end{aligned}$$

Obviously, $I_1 \le M\|(\varphi_1 - \varphi_2)(0)\|_\alpha$.

By using (F1) and (G1), one can verify that

$$\begin{aligned}I_2 &\le K_\alpha K_1(\rho^*)\int_0^t (t-s)^{-\alpha}\|x(s) - y(s)\|_\alpha ds. \\ I_3 &\le K_\alpha K_2(\rho^*)\int_0^t (t-s)^{-\alpha}[\int_{-r}^s \|h(s-\theta)\|_{L(X)}\|x(\theta) - y(\theta)\|_\alpha d\theta]ds \\ &\le K_\alpha K_2(\rho^*)\int_0^t (t-s)^{-\alpha}[\int_{-r}^0 \|h(s-\theta)\|_{L(X)}\|\varphi_1(\theta) - \varphi_2(\theta)\|_\alpha d\theta]ds\end{aligned}$$

$$+ K_\alpha K_2(\rho^*)\int_0^t (t-s)^{-\alpha}\Big[\int_0^s \|h(s-\theta)_{L(X)}\|x(\theta)-y(\theta)\|_\alpha d\theta\Big]ds$$

$$\le K_\alpha K_1(\rho^*)\Big(\sup_{0\le s\le T}\int_{-r}^0 \|h(s-\theta)\|_{L(X)}d\theta\Big)\Big(\int_0^t (t-s)^{-\alpha}ds\Big)\sup_{-r\le\theta\le 0}\|(\varphi_1-\varphi_2)(\theta)\|_\alpha$$

$$+ K_\alpha K_2(\rho^*)\Big(\sup_{0\le s\le T}\int_0^s \|h(s-\theta)\|_{L(X)}d\theta\Big)\Big(\int_0^t (t-s)^{-\alpha}\sup_{0\le\theta\le s}\|x(\theta)-y(\theta)\|_\alpha d\theta\Big)ds$$

$$\le K_\alpha K_1(\rho^*)\bar{h}\,\frac{T^{1-\alpha}}{1-\alpha}\|\varphi_1-\varphi_2\|_{C([-r,0];X_\alpha)}$$

$$+ K_\alpha K_2(\rho^*)\bar{h}\int_0^t (t-s)^{-\alpha}\|x_s-y_s\|_C\,ds.$$

Then, for $t\in[0,T]$,

$$\begin{aligned}\|x(t)-y(t)\|_\alpha \le\ & M\|(\varphi_1-\varphi_2)(0)\|_\alpha\\ &+ K_\alpha K_1(\rho^*)\bar{h}\,\frac{T^{1-\alpha}}{1-\alpha}\|\varphi_1-\varphi_2\|_{C([-r,0];X_\alpha)}\\ &+ K_\alpha K_1(\rho^*)\int_0^t (t-s)^{-\alpha}\|x(s)-y(s)\|_\alpha ds\\ &+ K_\alpha K_2(\rho^*)\bar{h}\int_0^t (t-s)^{-\alpha}\|x_s-y_s\|_C\,ds.\end{aligned}$$

By using the Gronwall's lemma with singularity and time lag, we get

$$\|x(t)-y(t)\|_\alpha \le M_1(\rho^*)\|\varphi_1-\varphi_2\|_{C([-r,0];X_\alpha)},$$

for all $t\in[0,T]$, and $M_1(\rho^*) = M + K_\alpha K_1(\rho^*)\bar{h}\,\frac{T^{1-\alpha}}{1-\alpha}$.

Choose $K(\rho^*) = \max\{M_1(\rho^*), 1\}$. Then

$$\|x(t)-y(t)\|_\alpha \le K(\rho^*)\|\varphi_1-\varphi_2\|_{C([-r,0];X_\alpha)},$$

for all $t\in[-r,T]$.

Since ρ^* depends on ρ then

$$\|x-y\|_{C([-r,T];X_\alpha)} \le K(\rho)\|\varphi_1-\varphi_2\|_{C([-r,0];X_\alpha)}.$$

□

Corollary 3.4.2. Assume that the hypotheses of Theorem 3.2.2 are satisfied.
Let $\varphi_0 \in C([-r,0];X_\alpha)$ and x_{φ_0} be the corresponding mild solution of the system (3.1.1). Then for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x_\varphi - x_{\varphi_0}\|_{C([-r,T];X_\alpha)} < \varepsilon,$$

provided that $\|\varphi-\varphi_0\|_{C([-r,0];X_\alpha)} < \delta$, x_φ is the mild solution on [-r, T] corresponding to $\varphi \in C([-r,0];X_\alpha)$.

Proof. Since $x_{\varphi_0} \in C([-r,T];X_\alpha)$ then there exists a constant $\rho^* > 0$ such that

$$\|x_{\varphi_0}\|_{C([-r,T];X_\alpha)} \le \rho^*.$$

Let $\varepsilon > 0$ be given. If $\varphi \in B(\varphi_0;1)$ then $\|\varphi\|_{C([-r,0];X_\alpha)} \le 1 + \|\varphi_0\|_{C([-r,0];X_\alpha)} \le 1+\rho \equiv \hat{\rho}$.
By Theorem 3.4.1, there exists a constant $K(\hat{\rho}) > 0$ such that

$$\|x_\varphi - x_{\varphi_0}\|_{C([-r,T];X_\alpha)} \le K(\hat{\rho})\|\varphi-\varphi_0\|_{C([-r,0];X_\alpha)}.$$

Choose $\delta = \min\{1, \frac{\varepsilon}{K(\hat{\rho})}\}$ which is positive.

Let $\varphi \in B(\varphi_0; \delta)$. Then $\varphi \in B(\varphi_0; 1)$. And

$$\begin{aligned} \|x_{\varphi} - x_{\varphi_0}\|_{C([-r,T];X_\alpha)} &\le K(\hat{\rho})\|\varphi - \varphi_0\|_{C([-r,0];X_\alpha)} \\ &< K(\hat{\rho})\delta \\ &\le K(\hat{\rho})\left(\frac{\varepsilon}{K(\hat{\rho})}\right) = \varepsilon. \end{aligned}$$

□

Theorem 3.4.3. Assume that hypotheses of Theorem 3.2.2 are satisfied. For any $\rho > 0$, if x, y are mild solutions of the system (3.1.1) on [-r, T] corresponding to h_1 and h_2, respectively, then there exists a constant $L(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \le L(\rho)\|h_1 - h_2\|_{L_1([0,T+r];L(X))},$$

provided $h_1, h_2 \in L_1([0, T+r]; L(X))$ with $\|h_1\|_{L_1([0,T+r];L(X))} \le \rho$ and $\|h_2\|_{L_1([0,T+r];L(X))} \le \rho$.

Proof. Firstly, we show that if z is a mild solution of the system (3.1.1) on [-r, T] and z corresponds to $h \in L_1([0, T + r]; L(X))$ with $\|h\|_{L_1([0,T+r];L(X))} \le \rho$, then z satisfies the inequality

$$\|z\|_{C([-r,T];X_\alpha)} \le \rho^*,$$

for a constant $\rho^* > 0$ depending on ρ. In fact as in the proof of Theorem 3.2.2, it follows that for $t \in [0, T]$, by the Gronwall's lemma with singularity and time lag, there exists a constant $M_1 > 0$ such that

$$\|z(t)\|_\alpha \le M_1,$$

for all $t \in [0, T]$. Set $\rho^* \equiv \max\{M_1, \|\varphi\|_{C([-r,0];X_\alpha)}\}$. We have

$$\|z\|_{C([-r,T];X_\alpha)} \le \rho^*,$$

Next, we show that there exists a constant $L(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \le L(\rho)\|h_1 - h_2\|_{L_1([0,T+r];L(X))}.$$

For $t \in [-r, 0]$, $\|x(t) - y(t)\|_\alpha = \|\varphi(t) - \varphi(t)\|_\alpha = 0$.

For $t \in [0, T]$,

$$\begin{aligned} \|x(t) - y(t)\|_\alpha &\le \int_0^t \|T(t-s)(f(x(s)) - f(y(s)))\|_\alpha ds \\ &\quad + \int_0^t \|T(t-s)[\int_{-r}^s \{h_1(s-\theta)g(x(\theta)) - h_2(s-\theta)g(y(\theta))\}d\theta]\|_\alpha ds \\ &= I_1 + I_2. \end{aligned}$$

Since f is locally Lipschitz in $x \in X_\alpha$, $\|x(t)\|_\alpha \le \rho^*$ and $\|y(t)\|_\alpha \le \rho^*$, $t \in [-r, T]$ then there exists a constant $K_1(\rho^*) > 0$ such that

$$I_1 \le K_\alpha K_1(\rho^*)\int_0^t (t-s)^{-\alpha} \|x(s) - y(s)\|_\alpha ds.$$

By a similar argument, there exists a constant $K_2(\rho^*) > 0$ such that

$$I_2 \le K_\alpha \int_0^t (t-s)^{-\alpha} [\int_{-r}^s \|h_1(s-\theta)(g(x(\theta)) - g(y(\theta)))\|_X d\theta]ds$$

$$
\begin{aligned}
&\quad + K_\alpha \int_0^t (t-s)^{-\alpha} \Big[\int_{-r}^{s} \|(h_1(s-\theta) - h_2(s-\theta))g(y(\theta))\|_X d\theta \Big] ds \\
&\leq K_\alpha K_2(\rho^*) \Big(\sup_{s\in[0,T]} \int_{-r}^{T} \|h_1(s-\theta)\|_{L(X)} d\theta \Big) \Big[\int_0^t (t-s)^{-\alpha} \sup_{0\leq\theta\leq s} \|x(\theta) - y(\theta)\|_\alpha ds \Big] \\
&\quad + K_\alpha K_2 \int_0^t (t-s)^{-\alpha} \Big[\int_{-r}^{s} \|h_1(s-\theta) - h_2(s-\theta)\|_{L(X)} (1 + \|x(\theta)\|_\alpha) d\theta \Big] ds \\
&\leq K_\alpha K_2(\rho^*) \bar{h}_1 \int_0^t (t-s)^{-\alpha} \|x_s - y_s\|_C ds \\
&\quad + K_\alpha K_2 \frac{T^{1-\alpha}}{1-\alpha} \Big(\sup_{0\leq s\leq T} \int_0^{s+r} \|(h_1 - h_2)(\theta)\|_{L(X)} d\theta \Big) (1 + \rho^*)
\end{aligned}
$$

$$\begin{aligned}\leq \quad & K_\alpha K_2 \frac{T^{1-\alpha}}{1-\alpha}(1+\rho^*)\|h_1 - h_2\|_{L_1([0,\,T+r];\,L(X))} \\ & + K_\alpha K_2(\rho^*)\,\overline{h}_1 \int_0^t (t-s)^{-\alpha}\ \|x_s - y_s\|_C\, ds.\end{aligned}$$

Then

$$\begin{aligned}\|x(t)-y(t)\|_\alpha \ \leq\ & a(\rho^*)\|h_1 - h_2\|_{L_1([0,\,T+r];\,L(X))} \\ & + b(\rho^*)\int_0^t (t-s)^{-\alpha}\ \|x(s)-y(s)\|_\alpha ds \\ & + c(\rho^*)\int_0^t (t-s)^{-\alpha}\ \|x_s - y_s\|_C ds,\end{aligned}$$

where $a(\rho^*) = K_\alpha K_2 \frac{T^{1-\alpha}}{1-\alpha}(1+\rho^*)$, $b(\rho^*) = K_\alpha K_1(\rho^*)$, $c(\rho^*) = K_\alpha K_2(\rho^*)\,\overline{h}_1$.

By the Gronwall's lemma with singularity and time lag and ρ^* as depends on ρ, there exists a constant $M_1 > 0$ such that

$$\begin{aligned}\|x(t)-y(t)\|_\alpha \ \leq\ & M_1 a(\rho^*)\|h_1 - h_2\|_{L_1([0,\,T+r];\,L(X))} \\ =\ & L(\rho)\|h_1 - h_2\|_{L_1([0,\,T+r];\,L(X))}.\end{aligned}$$

for all $t \in [0, T]$, where $L(\rho) = M_1 a(\rho^*) = M_1 a(\rho)$, hence this inequality holds for $t \in [-r, T]$.

Therefore

$$\|x - y\|_{C([-r,T];X_\alpha)} \ \leq\ L(\rho)\|h_1 - h_2\|_{L_1([0,\,T+r];\,L(X))}. \qquad \square$$

Corollary 3.4.4. Assume that hypotheses of Theorem 3.2.2 are satisfied.

Let $h_0 \in L_1([0, T + r]; L(X))$ and x_{h_0} be the mild solution of the system (3.1.1) corresponding to h_0. Then for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$\|x_h - x_{h_0}\|_{C([-r,T];X_\alpha)} < \varepsilon,$$

provided that $\|h - h_0\|_{L_1([0,T+r];L(X))} < \delta$, where x_h is the mild solution of the system (3.1.1) corresponding to h. That is, the operator $H : L_1([0, T + r]; L(X)) \to C([-r, T]; X_\alpha)$, defined by $H(h) = x_h$ is continuous.

Proof. Let $\varepsilon > 0$. Since $h_0 \in L_1([0, T + r]; L(X))$ then $\|h_0\|_{L_1([0,T+r];L(X))} < \rho$ for a constant $\rho > 0$. By Theorem 3.4.3 we get that $\|x_{h_0}\|_{C([-r,T];X_\alpha)} \leq \rho^*$ for a constant $\rho^* > 0$. If $h \in L_1([0, T + r]; L(X))$ and $\|h - h_0\|_{L_1([0,T+r];L(X))} < 1$, it follows that

$$\|h\|_{L_1([0,T+r];L(X))} \leq 1 + \|h_0\|_{L_1([0,T+r];L(X))} < 1 + \rho = \hat{\rho}.$$

By Theorem 3.4.3 again, there exists a constant $L(\hat{\rho}) > 0$ such that

$$\|x_h - x_{h_0}\|_{C([-r,T];X_\alpha)} \leq L(\hat{\rho})\|h - h_0\|_{L_1([0,T+r];L(X))}.$$

Choose $\delta = \min\{1, \frac{\varepsilon}{L(\hat{\rho})}\}$.

Let $h \in B(h_0; \delta)$. Then $h \in B(h_0; 1)$, and

$$\begin{aligned} \| x_h - x_{h_0} \|_{C([-r,T];X_\alpha)} &\le L(\hat{\rho}) \| h - h_0 \|_{L_1([0,T+r];L(X))} \\ &\le L(\hat{\rho})\,\delta \\ &\le L(\hat{\rho}) \left(\frac{\varepsilon}{L(\hat{\rho})} \right) = \varepsilon. \end{aligned}$$ □

Corollary 3.4.5. Assume that hypotheses of Theorem 3.2.2 are satisfied.

If $x \in C([-r,T];X_\alpha)$ is a mild solution of the system (3.1.1) on [0, T] with $\varphi \in C([-r,0];X_\alpha)$ and $h \in L_1([0,T+r];L(X))$, define $G(\varphi, h) = x$. Then the operator $G : C([-r,0];X_\alpha) \times L_1([0,T+r];L(X)) \to C([-r,T];X_\alpha)$ is continuous.

Proof. Let (φ_n) be a sequence in $C([-r,0];X_\alpha)$ such that $\varphi_n \to \varphi$ in $C([-r,0];X_\alpha)$. Let (h_n) be a sequence in $L_1([0,T+r];L(X))$ such that $h_n \to h$ in $L_1([0,T+r];L(X))$.

For each n, let x_n be a mild solution of the system (3.1.1) corresponding to φ_n and h_n.

Without loss of generality, we can assume that $\|\varphi\|_{C([-r,0];X_\alpha)}$, $\|\varphi_n\|_{C([-r,0];X_\alpha)}$ and $\|h_n\|_{L_1([0,T+r];L(X))} \le \rho_1$, for a constant $\rho_1 > 0$.

There exists a constant $\rho_2 > 0$ such that $\|x_n\|_{C([-r,T];X_\alpha)} \le \rho_2$. Set $\rho = \max\{\rho_1, \rho_2\}$.

By Theorem 3.4.1 and Theorem 3.4.3, there are constants $K(\rho)$ and $L(\rho)$ such that

$$\| x_n - x \|_{C([-r,T];X_\alpha)} \le K(\rho)\, \| \varphi_n - \varphi \|_{C([-r,0];X_\alpha)} + L(\rho)\, \| h_n - h \|_{L_1([0,T+r];L(x))}.$$

Since $\varphi_n \to \varphi$ and $h_n \to h$ then $x_n \to x$ in $C([-r,T];X_\alpha)$.

So G is continuous on $C([-r,0];X_\alpha) \times L_1([0,T+r];L(X))$. The proof is complete. □

3.5 A Semilinear System with Infinite Delay

Consider the following semilinear integrodifferential equation with infinite delay

$$\begin{cases} \dfrac{d}{dt} x(t) + Ax(t) = f(t, x(t)) + \displaystyle\int_{-\infty}^{t} h(t-s) g(s, x(s)) ds, \; t \in [0,T], \\ \qquad x(t) = \varphi(t), \; t \in (-\infty, 0]. \end{cases} \tag{3.5.1}$$

Let $BC((-\infty,T];X_\alpha)$ denote the Banach space of all bounded continuous X_α – valued functions defined on $(-\infty, T]$, with the sup-norm. For a fixed $\varphi \in BC((-\infty,0];X_\alpha)$, let C_φ denote $\{x \in C([0,T];X_\alpha) \mid x(0) = \varphi(0)\}$. Then C_φ is a nonempty closed convex subset of $C([0,T];X_\alpha)$.

We investigate the existence problem to the system (3.5.1). To obtain local existence of mild solutions, we impose the following assumptions.

Assumptions

(G7) The function $g : (-\infty, T] \times X_\alpha \to X$ satisfies

(i) $g(\bullet, x)$ is measurable on $(-\infty, T]$, for each $x \in X_\alpha$

(ii) $g(t, \bullet)$ is locally Lipschitz continuous in X_α, for all $t \in (-\infty, T]$, i. e., for any $t \in (-\infty, T]$ and any $\rho > 0$, there exists a constant $K_2(t, \rho) > 0$ such that

$$\| g(s, x_1) - g(s, x_2) \|_X \leq K_2(t, \rho) \| x_1 - x_2 \|_\alpha,$$

for all $s \in (-\infty, t]$ and $x_1, x_2 \in X_\alpha$ such that $\|x_1\|_\alpha \leq \rho$ and $\|x_2\|_\alpha \leq \rho$.

(iii) g maps every bounded set in $(-\infty, T] \times X_\alpha$ to a bounded set in X.

(H3) $h \in L_1([0, \infty); L(X))$.

Definition 3.5.1. A function $x \in C((-\infty, a]; X_\alpha)$, $a \in (0, T]$, is called a *mild* solution of the system (3.5.1) if it satisfies the integral equation

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_{-\infty}^s h(s-\theta)g(\theta, x(\theta))d\theta]ds, t \in [0, a], \\ \varphi(t), t \in (-\infty, 0]. \end{cases} \tag{3.5.2}$$

Theorem 3.5.2. Assume that (A), (F3), (F5), (G5), (G7), (H3) hold. Let $\varphi \in BC((-\infty, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.5.1) has a unique mild solution $x \in C((-\infty, T]; X_\alpha)$.

Proof. Let $\varphi \in BC((-\infty, 0]; X_\alpha)$. Define an operator G on C_φ by

$$(Gx)(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_{-\infty}^s h(s-\theta)g(\theta, \tilde{x}(\theta))d\theta]ds, t \in [0, T],$$

where $\tilde{x}(t) = \begin{cases} x(t), t \in [0, T], \\ \varphi(t), t \in (-\infty, 0]. \end{cases}$ (3.5.3)

By a similar argument as in Lemma 3.1.3, one can show that $G : C_\varphi \to C_\varphi$.

As in the proof of Theorem 3.1.4, there exists a positive number t_1 depending only on φ, and a nonempty closed convex set B subset of C_φ defined by $B = \{ \xi \in C_\varphi \mid \| \xi(t) - \varphi(0) \|_\alpha \leq 1, t \in [0, t_1]\}$ such that $G : B \to B$ is a contraction.

By the Contraction mapping theorem, G has a unique fixed point x in B.

As in Theorem 3.2.2, applying the growth condition (F5) and (G5) and Lemma 3.1.6, one shows that if y is a mild solution of the system (3.5.1) on a subset $(-\infty, T']$, it follows that there exists a constant $\rho > 0$ such that

$$\|y(t)\|_\alpha \leq \rho,$$

for any $t \in (-\infty, T']$.

By using this a priori estimate, one can obtain interval of existence with equal length $\delta > 0$, $[t_1, t_1 + \delta]$, $[t_1 + \delta, t_1 + 2\delta]$, …, $[t_1 + n\delta, t_1 + (n+1)\delta]$, …, so that $T \in [t_1 + n\delta, t_1 + (n+1)\delta]$ for an

$n \in \mathbb{N}, \delta$ depends only on ρ. Hence the system (3.5.1) has a unique global mild solution on $(-\infty, T]$. □

Chapter IV

Optimal Control

In this chapter, we study existence of a control for a controlled system with finite delay. Existence of optimal control for a more general controlled system is investigated. We also study Bolza optimal control problem.

4.1 A Controlled System with Finite Delay

Consider the controlled system with finite delay:

$$\begin{cases} \frac{d}{dt}x(t) + Ax(t) = f(t, x(t)) + \int_{-r}^{t} h(t-s)g(s, x(s))ds + Bu(t), \\ x(t) = \varphi(t), t \in [-r, 0]. \end{cases} \tag{4.1.1}$$

We intend to use main results in the chapter III; especially Theorem 3.2.3, and apply to the controlled system (4.1.1) corresponding to the system (3.1.9). Here we impose some assumptions that are suitable to guarantee the existence of mild solutions of the controlled system (4.1.1).

Assumptions

(A1) X is a separable reflexive Banach space. -A is the infinitesimal generator of an analytic semigroup T(t), $t \geq 0$ on the Banach space X.

(B) E is a reflexive Banach space which the controls u take their values and $B \in L(L_p(I, E), L_p(I, X))$, where $I \equiv [0, T]$.

Definition 4.1.1. For any $u \in L_p(I, E)$ and any $\varphi \in C([-r,0]; X_\alpha)$, if there exists a constant $t_0 = t_0(u, \varphi) > 0$ and $x \in C([-r, t_0]; X_\alpha)$ such that

$$x(t) = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(s, x(s))ds + \int_0^t T(t-s)[\int_{-r}^{s} h(s-\theta)g(\theta, x(\theta))d\theta]ds \\ \qquad + \int_0^t T(t-s)Bu(s)ds, t \in [0, t_0], \\ \varphi(t), t \in [-r, 0]. \end{cases} \tag{4.1.2}$$

then the system (4.1.1) is called *mildly solvable* with respect to u on $[-r, t_0]$, and $x \in C([-r, t_0]; X_\alpha)$ is said to be an α–mild solution with respect to u on $[-r, t_0]$.

Theorem 4.1.2. Suppose the assumptions (A1), (B), (F2), (F6), (G2), (G6) and (H1) hold. Let u $\in$ L_p(I, E), p > $\frac{1}{1-\alpha}$, $\varphi \in C([-r,0];X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (4.1.1) is mildly solvable on [–r, T] with respect to u, and the α–mild solution is unique.

Proof. By using corollary 2.4.23 and Theorem 3.2.3, it is sufficient to prove that

$v(t) = \int_0^t T(t-s)Bu(s)ds$ is continuous on [0, T].

Suppose $0 \le t_1 < t_2 \le T$. Then

$$\begin{aligned}
\| v(t_2) - v(t_2) \|_\alpha &\le \| \int_0^{t_2} T(t_2 - s)Bu(s)ds - \int_0^{t_1} T(t_1 - s)Bu(s)ds \|_\alpha \\
&\le \int_0^{t_1} \| [T(t_2 - s) - T(t_1 - s)]Bu(s) \|_\alpha \, ds + \int_{t_1}^{t_2} \| T(t_2 - s)Bu(s) \|_\alpha \, ds \\
&\le \int_0^{t_1} \| (T(t_2 - t_1) - I)T(t_1 - s)Bu(s) \|_\alpha \, ds + K_\alpha \int_{t_1}^{t_2} (t_2 - s)^{-\alpha} \| Bu(s) \|_X \, ds \\
&= I_1 + I_2.
\end{aligned}$$

Since $\alpha < \beta \le 1$, by using Theorem 2.3.16(c),(d) and Hölder's inequality, it follows that

$$I_1 \le C_{\beta-\alpha} (t_2 - t_1)^{\beta-\alpha} \frac{t_1^{\beta-\alpha}}{1-\beta q} \| Bu \|_{L_p(I,X)} .$$

$$I_2 \le K_\alpha \frac{(t_2 - t_1)^{1-\alpha q}}{1-\alpha q} \| Bu \|_{L_p(I,X)} .$$

These inequalities yield that v is continuous on [0, T]. ☐

We will now study a system that is more general than system (4.1.1). We investigate the existence of mild solutions of the controlled system. We impose some assumptions that is sufficient to guarantee existence of mild solutions.

Assumptions

(A2) The function f : [0, T] × X_α × E → X satisfies

(i) f(·, x, u) is continuous on [0,T], for each x $\in$ X_α and each u $\in$ E.

(ii) f(t, ·, ·) is continuous on X_α × E, for a. e. t $\in$ [0, T].

(iii) f(t, ·, u) is locally Lipschitz continous on X_α, for a. e. t $\in$ [0, T] and each u $\in$ E, i. e., for a. e. t $\in$ [0, T]and any ρ ≥ 0 there exists a constant $K_1(t, \rho, u) > 0$ such that

$$\|f(s, x_1, u) - f(s, x_2, u)\| \le K_1(t, \rho, u)\|x_1 - x_2\|_\alpha,$$

for all $s \in [0,t]$ and $\|x_1\|_\alpha \le \rho$ and $\|x_2\|_\alpha \le \rho$.

(A3) The function g : [0, T] × X_α × E → X satisfies

(i) g(·, x, u) is continuous on [0, T], for each x $\in$ X_α and each u $\in$ E.

(ii) $g(t, \cdot, \cdot)$ is continuous on $X_\alpha \times E$, for a. e. $t \in [0, T]$.

(iii) $g(t, \cdot, u)$ is locally Lipschitz continous on X_α, for a. e. $t \in [0, T]$ and each $u \in E$, i. e., for a. e. t in [0, T] , for each u in E and any $\rho \geq 0$ there exists a constant $K_2(\rho, u) > 0$ such that

$$||g(s, x_1, u) - g(s, x_2, u)||_X \leq K_1(\rho, u)||x_1 - x_2||_\alpha,$$

for all $s \in [0, t]$ and $|| x_1 ||_\alpha \leq \rho$ and $|| x_2 ||_\alpha \leq \rho$.

(H) $h \in L_1([0, T]; L(X))$.

We consider the following controlled system

$$\begin{cases} \frac{d}{dt} x(t) + Ax(t) = f(t, x(t), u(t)) + \int_0^t h(t-s)g(s, x(s), u(s))ds, \; t \in [0, T], \\ \qquad x(0) = x_0, \end{cases} \tag{4.1.4}$$

where $u \in U_{ad}$ (= the admissible control set a nonempty closed convex bounded subset of $L_p(I, E)$).

Definition 4.1.3. For every $u \in L_p(I, E)$, if there exists a $t_0 = t_0(u) > 0$ and $x \in C([0, t_0]; X_\alpha)$ such that

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s), u(s))ds + \int_0^t T(t-s)[\int_0^s h(s-\theta)g(\theta, x(\theta), u(\theta))d\theta]ds, \; 0 \leq t \leq T, \tag{4.1.5}$$

then the system (4.1.4) is called *mildly solvable* with respect to u on $[0, t_0]$ and $x \in C([0, t_0]; X_\alpha)$ is said to be an α–mild solution with respect to u.

Theorem 4.1.4. Assume that assumptions (A1), (A2), (A3) and (H) hold. Then for each $u \in U_{ad}$ and each $x_0 \in X_\beta$ for some $\beta \in (\alpha, 1]$, there exists a constant $t_0 = t_0(u) > 0$ such that the controlled system (4.1.4) is mildly solvable on $[0, t_0]$ with respect to u, and the α–mild solution is unique.

Proof. Let $u \in U_{ad}$. Since u is fixed, define

$$\tilde{f}(t, x) = f(t, x, u(t)),$$

$$\tilde{g}(t, x) = g(t, x, u(t)),$$

for $t \in [0, T]$ and $x \in X_\alpha$.

We show that $\tilde{f}$ and $\tilde{g}$ satisfy (F2) and (G2), respectively.

Since $f(\bullet, x, u(\bullet))$ and $g(\bullet, x, u(\bullet))$ are continuous on [0, T] for each $x \in X_\alpha$ and each $u \in L_p(I, E)$, then $\tilde{f}(\bullet, x)$ and $\tilde{g}(\bullet, x)$ are continuous on [0, T].

Similarly, since $f(t, \bullet, u(t))$ and $g(t, \bullet, u(t))$ are locally Lipschitz on X_α, then $\tilde{f}(t, \bullet)$ and $\tilde{g}(t, \bullet)$ are locally Lipschitz on X_α. Thus $\tilde{f}$ and $\tilde{g}$ satisfy (F2) and (G2), respectively.

Since u is fixed, by Theorem 3.1.8, there exists a constant $t_0 = t_0(x_0, u) > 0$ such that the system (4.1.4) has a unique mild solution on $[0, t_0]$. Therefore the system (4.1.4) is mildly solvable on $[0, t_0]$. □

4.2 Existence of Optimal Controls

In the following we consider a Bolza optimal control problem for the controlled system (4.1.1). Under the assumptions of Theorem 4.1.2, for each fixed $u \in L_p(I, E)$, the system (4.1.1) is mildly solvable on $I = [0, T]$.

Let U_{ad} be the admissible control subset of $L_p(I, E)$. We consider the Bolza problem (P), i. e.,

(P) : Find $u^\circ \in U_{ad}$ such that $J(u^\circ) \le J(u)$, for all $u \in U_{ad}$, where

$$J(u) = \int_I \ell(t, x^u(t), u(t))dt + \psi(x^u(T)),$$

where x^u denotes the mild solution of system (4.1.1) corresponding to the control $u \in U_{ad}$, and $\psi : X_\alpha \to \nabla$ is a nonnegative continuous function.

$\{u, x^u\}$ is called an admissible state-control pair, or simply admissible pair. For the existence of a solution of the Bolza problem (P) we shall introduce the following assumptions:

(U) $U_{ad} = L_p(I, E)$, $B \in L(L_p(I, E), L_p(I, X))$, $1 < p < \infty$, and B is strongly continuous.

(L) The function ℓ: $I \times X_\alpha \times E \to \nabla \cup \{\infty\}$ is Borel measurable satisfying the following conditions:

1. $\ell(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X_\alpha \times E$ for almost all $t \in I$.
2. $\ell(t, x, \cdot)$ is convex on E for each $x \in X_\alpha$ and almost all $t \in I$.
3. There exists constants $b \ge 0$, $c > 0$ and $\phi \in L_1(I, \nabla)$ such that

$$\ell(t, x, u) \ge \phi(t) + b\| x \|_\alpha + c\| u \|_E^P,$$

for all $t \in I$.

(ψ) The function $\psi : X_\alpha \to \nabla$ is continuous and nonnegative.

We refer to a remarkable result about strong-weak lower semicontinuity of a functional, Balder gives this result in his paper (See Balder, E. J. (1987)). The result is

Theorem 4.2.1. (Balder's Theorem)

Let $(X, \| \bullet \|)$ be a separable Banach space, and $(V, | \bullet |)$ a separable reflexive Banach space, whose dual is denoted by V'. Let $\ell : I \times X \times V \to (-\infty, +\infty]$ be a given measurable function.

The following three conditions

$$\ell(t,\bullet,\bullet) \text{ is sequentially l.s.c. on } X \times V \ \mu\text{-a. e.,}$$

$$\ell(t,x,\bullet) \text{ is convex on V for every } x \in X \ \mu\text{-a. e.,}$$

there exist $M > 0$ and $\phi \in L_1(\nabla)$ such that

$$\ell(t,x,v) \geq \phi(t) - M(\|x\| + |v|) \text{ for all } x \in X,\ v \in V \mu \text{ a. e.,}$$

are sufficient for sequential strong-weak lower semicontinuity of I_ℓ on $L_1(X) \times L_1(V)$. Moreover, they are also necessary, provided that $I_\ell(\bar{x},\bar{v}) < +\infty$ for some $\bar{x} \in L_1(X), \bar{v} \in L_1(V)$, where $I_\ell : L_1(X) \times L_1(V) \to [-\infty, +\infty]$ is the associated integral functional defined by

$$I_\ell(x,v) \equiv \int_I \ell(t,x(t),v(t))\mu dt.$$

Proof. See Balder, E. J. (1987), pp. 1399-1404.

We now present the main theorem for the Bolza problem.

Theorem 4.2.2. Suppose the assumptions (A1), (B), (F2), (G2), (F6), (G6), (H1), (U), (L) and (ψ) hold. Let $\varphi \in C([-r,0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the Bolza problem (P) has a solution, i. e., there exists an admissible state-control pair $\{u^\circ, x^\circ\}$ such that

$$J(u^\circ) = \int_I \ell(t, x^0(t), u^0(t))dt + \psi(x^0(T)) \leq J(u), \text{ for all } u \in U_{ad}.$$

Proof: If $\inf \{J(u) \mid u \in U_{ad}\} = +\infty$, there is nothing to prove.

Assume that $\inf\{J(u) \mid u \in U_{ad}\} = m < \infty$

By (L)–3, there exists constants $b \geq 0$, $c > 0$ and $\phi \in L_1(I, \nabla)$ such that

$$\ell(t, x, u) \geq \phi(t) + b\|x\|_\alpha + c\|u\|_E^p.$$

Then

$$\begin{aligned} J(u) = & \int_I \ell(t,x(t),u(t))dt + \psi(x^u(T)) \\ \geq & \int_I \phi(t)dt + b\int_I \|x(t)\|_\alpha dt + c\int_I \|u(t)\|_E^p dt + \psi(x^u(T)) \\ \geq & -\eta \\ > & -\infty, \end{aligned}$$

where $\eta > 0$ is a constant. Hence $m \geq -\eta > -\infty$.

By the definition of infimum, there exists a minimizing sequence $\{u^n\}$ of J, i. e., $J(u^n) \to m$ as $n \to \infty$. By the assumption (L)-3 again, we have

$$\ell(t, x, u^n) \geq \phi(t) + b\|x\|_\alpha + c\|u^n\|_E^p.$$

Then

$$J(u^n) \geq \int_I \phi(t)dt + b\int_I \|x(t)\|_\alpha dt + c\int_I \|u(t)\|_E^p dt + \psi(x^u(T)).$$

So

$$m - \int_I \phi(t)dt - b\int_I \| x(t) \|_\alpha \, dt - \psi(x^u(T)) \geq c\| u^n \|_{L_p(I,E)} .$$

Therefore $\| u^n \|_{L_p(I, E)} \leq m_1/c$ for all n, for a constant m_1 independent of n.

This shows that $\{u^n\}$ is contained in a bounded subset of the reflexive Banach space $L_p(I, E)$. So $\{u^n\}$ has a subsequence relabeled as $\{u^n\}$ and there is an element $u^\circ \in U_{ad}$ such that $u^n \xrightarrow{w} u^\circ$ in $L_p(I, E)$. Let $\{x^n\} \subset C([-r, T]; X_\alpha)$ denote the corresponding sequence of solutions for the integral equation

$$\begin{cases} x^n(t) = T(t)\varphi(0) + \int_0^t T(t-s)Bu^n(s)ds + \int_0^t T(t-s)f(s, x^n(s))ds \\ \qquad\qquad + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta)g(\theta, x^n(\theta))d\theta]ds, \; t \in [0, T], \\ x^n(t) = \varphi(t), \; t \in [-r, 0]. \end{cases}$$

Since $\| u^n \|_{L_p(I,E)}$ is bounded, by a similar argument to obtaining an a priori estimate as in Theorem 3.2.3, there exists a constant $\rho > 0$ such that

$$\| x^n \|_{C([0, T]; X_\alpha)} \leq \rho , \text{ for all } n = 0, 1, 2, \ldots .$$

where x^0 denotes the solution corresponding to u°, that is,

$$\begin{cases} x^\circ(t) = T(t)\varphi(0) + \int_0^t T(t-s)Bu^\circ(s)ds \\ \qquad\qquad + \int_0^t T(t-s)f(s, x^\circ(s))ds + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta)g(\theta, x^\circ(\theta))d\theta]ds, \; t \in [0, T], \\ x^\circ(t) = \varphi(t), \; t \in [-r, 0]. \end{cases}$$

By assumptions (F2) and (G2), for each t in [0, T], there exists positive constants $K_1(t, \rho)$, $K_2(t, \rho)$ such that

$$\|f(s, x^n(s)) - f(s, x^\circ(s))\| \leq K_1(t, \rho)\|x^n(s) - x^\circ(s)\|_\alpha, \; s \in [0, t],$$

and

$$\|g(\theta, x^n(\theta)) - g(\theta, x^\circ(\theta))\| \leq K_2(t, \rho)\|x^n(\theta) - x^\circ(\theta)\|_\alpha, \; \theta \in [-r, t].$$

Hence

$$\begin{aligned} \|x^n(t) - x^\circ(t)\|_\alpha \leq \; & \| \int_0^t T(t-s)B(u^n(s) - u^\circ(s))ds \\ & + \int_0^t T(t-s)[f(s, x^n(s)) - f(s, x^\circ(s))]ds \\ & + \int_0^t T(t-s)[\int_{-r}^s h(s-\theta)(g(\theta, x^n(\theta)) - g(\theta, x^\circ(\theta)))d\theta]ds\|_\alpha \end{aligned}$$

$$\leq K_\alpha \int_0^t (t-s)^{-\alpha} \|Bu^n(s) - Bu^\circ(s)\|_X ds$$

$$+ K_\alpha K_1(t, \rho) \int_0^t (t-s)^{-\alpha} \|x^n(s) - x^0(s)\|_\alpha \, ds$$

$$+ K_\alpha K_2(t, \rho) \int_0^t (t-s)^{-\alpha} [\int_{-r}^{s} \|h(s-\theta)\|_{L(X)} \, d\theta] \sup_{0\leq\theta\leq s} \|x^n(\theta) - x^0(\theta)\|_\alpha \, ds$$

$$\leq K_\alpha \frac{T^{1-\alpha q}}{1-\alpha q} \|Bu^n - Bu^\circ\|_{L_p(I, X)}$$

$$+ K_\alpha K_1(t, \rho) \int_0^t (t-s)^{-\alpha} \|x^n(s) - x^0(s)\|_\alpha \, ds$$

$$+ K_\alpha K_2(t, \rho) \bar{h} \int_0^t (t\text{-}s)^{-\alpha} \|x_s^n - x_s^0\|_C ds.$$

By Gronwall's lemma with singularity and time lag, $\|x^n(t) - x^\circ(t)\|_\alpha \leq M\|Bu^n - Bu^\circ\|_{L_p(I,\, X)}$, where $M = K_\alpha \frac{T^{1-\alpha q}}{1-\alpha q}$ is a constant, independent of n.

Since B is strongly continuous, we have $\|Bu^n - Bu^\circ\|_{L_p(I,\, X)} \xrightarrow{s} 0$ as $n \to \infty$. This implies $\|x^n - x^\circ\| \xrightarrow{s} 0$ in $C([-r, T]; X_\alpha)$.

The assumption (L) implies the assumption of Balder's theorem. Hence by the Balder's result, $(u, x) \to \int_I \ell(t, x^u(t), u(t))dt$ is sequential strong-weak lower semicontinuous on $L_1(I, E) \times L_1(I, X)$. Then J is weakly lower semicontinuous on $L_p(I, E)$. By (L)-3, since $J > -\infty$, J attains its minimum at $u^\circ \in U_{ad}$. Therefore the Bolza optimal control Problem (P) has a solution. □

Chapter V

Applications

In this chapter, we present some examples that illustrate our abstract results. These examples deal with controll problems subject to a class of semilinear evolution equations with delay. We apply Theorem 4.1.2 and Theorem 4.2.2 to prove the existence of an optimal control.

The first part of this chapter is about basic concepts of Sobolev spaces, strongly elliptic operators and related results. The second part consists of our examples that we introduce constructively to show how our abstract results can be applied.

5.1 Terminology

In the following we use $y = (y_1, y_2, \ldots, y_n)$ to be a variable point in the n-dimensional Euclidean space ∇^n. For any two such points $y = (y_1, y_2, \ldots, y_n)$ and $z = (z_1, z_2, \ldots, z_n)$ we set $y \cdot z = \sum_{i=1}^{n} y_i z_i$ and $|y|^2 = y \cdot y$.

An n-tuple of nonnegative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called a multi-index and we define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i$$

and

$$y^{\alpha} = y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \text{ for } y = (y_1, y_2, \ldots, y_n).$$

Denoting $D_k = \partial / \partial y_k$ and $D = (D_1, D_2, \ldots, D_n)$ we have

$$D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial y_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial y_n^{\alpha_n}}.$$

Let Ω be a fixed domain in ∇^n with boundary and closure $\overline{\Omega}$. Assume that $\partial\Omega$ is sufficiently smooth, e. g., $\partial\Omega$ is of the class C^k for some suitable $k \geq 0$, this means that for each point $y \in \partial\Omega$ there is a ball B with center at y such that $\partial\Omega \cap B$ can be represented in the form $y_i = \varphi(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ for some i, and φ is a k-times continuously differentiable function.

For a nonnegative integer m, we denote by $C^m(\Omega)$ (resp. $C^m(\overline{\Omega})$) the set of all m-times continuously differentiable real-valued or complex-valued functions in Ω (resp. $\overline{\Omega}$), by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ consisting of those functions which have compact support in Ω.

For $x \in C^m(\Omega)$ and $1 \leq p < \infty$, we define

$$\| x \|_{m,p} = \left(\int_{\Omega} \sum_{|\alpha|\le m} | D^{\alpha} x |^{p} \, dy \right)^{\frac{1}{p}} . \qquad (5.1.1)$$

Also for $p = 2$ and $u, v \in C^m(\Omega)$, we define

$$(u, v)_m = \int_{\Omega} \sum_{|\alpha|\le m} D^{\alpha} u \overline{D^{\alpha} v} \, dy . \qquad (5.1.2)$$

Let $\hat{C}_p^m(\Omega)$ be the subset of $C^m(\Omega)$ consisting of those functions x for which $\|x\|_{m,\,p} < \infty$. We define $W^{m,\,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ to be the completions in the norm $\| \bullet \|_{m,p}$ of $\hat{C}_p^m(\Omega)$ and $C_0^m(\Omega)$, respectively. The spaces $W^{m,\,p}(\Omega)$ consists of functions $x \in L^p(\Omega)$ whose derivatives $D^{\alpha} x$ in the sense of distributions, of order $|\alpha| \le m$ are in $L^p(\Omega)$, and $W_0^{m,p}(\Omega)$ is the closure of $C_0^m(\Omega)$ in $W^{m,\,p}(\Omega)$.

It is well known that $W^{m,\,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces with the usual norm $\| \bullet \|_{m,p}$. Then $W^{m,\,p}(\Omega)$ is separable, uniformly convex and hence reflexive. Let

$$H^m(\Omega) = W^{m,\,2}(\Omega), \ H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces with the scalar product $(\cdot,\cdot)$ given by (5.1.2). The following imbedding theorem describes various relations among the above spaces.

Theorem 5.1.1. (Sobolev) The following relations among $W^{m,\,p}(\Omega)$, $C^m(\Omega)$, and $L^p(\Omega)$ hold:

(1) $W^{m,\,p}(\Omega) \subset W^{m,\,r}(\Omega)$ if $1 \le r \le p$, and the imbedding is continuous;

(2) $W^{m,\,r}(\Omega) \subset W^{j,\,p}(\Omega)$ if $1 \le r, p < \infty$, j and m are integers such that $0 \le j < m$ and $\frac{1}{p} > \frac{1}{r} + \frac{j}{n} - \frac{m}{n}$, and the imbedding is compact;

(3) $W^{m,\,p}(\Omega) \subset L^{\frac{np}{n-mp}}(\Omega)$ if $mp < n$ and there exists a constant c_1 such that

$$\| x \|_{0,\frac{np}{n-mp}} \le c_1 \| x \|_{m,p}, \text{ for } x \in W^{m,\,p}(\Omega);$$

(4) $W^{m,\,p}(\Omega) \subset C^k(\overline{\Omega})$ if $0 \le k < m - \frac{n}{p}$, and there exists a constant c_2 such that

$$\sup\{| D^{\alpha} x(y) |;\ |\alpha| \le k, y \in \overline{\Omega}\} \le c_2 \|x\|_{m,\,p}, \text{ for } x \in W^{m,\,p}(\Omega);$$

(5) (Poincaré Inequality) There exists a constant $c = c(\Omega)$ such that

$$\inf_{k\in R} \| x + k \|_{0,2} \le c(\Omega) \| \nabla x \|_{0,2}, \text{ for } x \in H_0^1(\Omega).$$

Since $\partial\Omega$ is smooth, $C_0^{\infty}(\Omega)$ is dense in $W_0^{m,p}(\Omega)$ and $L_2(\Omega)$, $W_0^{m,p}(\Omega)$ is dense in $L_2(\Omega)$. From Sobolev's imbedding theorem, we have that the imbeddings

$$C_0^{\infty}(\Omega) \hookrightarrow W_0^{m,p}(\Omega) \hookrightarrow L_2(\Omega).$$

For any $\sigma = k + \eta > 0$, where k is a nonnegative integer and $\eta \in (0, 1)$, $C^{\sigma}(\overline{\Omega})$ denotes the Banach space consisting of those functions belonging to $C^{k}(\overline{\Omega})$ whose derivatives $D^{\alpha}x$ of order $|\alpha| = k$ satisfy a uniform Hölder condition with exponent η. The norm in this space is defined as

$$\|x\|_{C^{\sigma}(\overline{\Omega})} = \|x\|_{C^{k}(\overline{\Omega})} + \sum_{|\alpha|=k} [D^{\alpha}x]_{\eta},$$

with

$$[\upsilon]_{\eta} = \sup_{y,z\in\Omega, y\neq z} \frac{|\upsilon(y)-\upsilon(z)|}{|y-z|^{\eta}}.$$

For a bounded domain Ω in R^{n} with a smooth boundary $\partial\Omega$, we consider the differential operator of order 2m,

$$A(y, D) = \sum_{|\alpha|\leq 2m} a_{\alpha}(y)D^{\alpha} \tag{5.1.3}$$

where the coefficients $a_{\alpha}(y)$ are sufficiently smooth complex-valued functions of y in $\overline{\Omega}$. The principal part $A'(y, D)$ of A(y, D) is the operator

$$A'(y, D) = \sum_{|\alpha| = 2m} a_{\alpha}(y)D^{\alpha} \tag{5.1.4}$$

Definition 5.1.2. The operator A(y, D) is strongly elliptic if there exists a constant $c > 0$ such that

$$\operatorname{Re}(-1)^{m} A'(y, \xi) \geq c|\xi|^{2m}, \tag{5.1.5}$$

for all $y \in \overline{\Omega}$ and $\xi \in R^{n}$.

For example the Laplacian operator Δ given by

$$\Delta x = \sum_{i=1}^{n} \frac{\partial^{2}x}{\partial y_{i}^{2}}.$$

$-\Delta$ is clearly strongly elliptic.

5.2 Optimal Control of a Semilinear System with Finite Delay

In the following, we give some examples of infinitesimal generator of analytic semigroup in Example1and the existence of an optimal control for a semilinear parabolic controlled system with finite delay in Example2. It is important to know which differential operator can be the infinitesimal generator of an analytic semigroup. We collect some important generators as follows.

Example 1.

Let $A(y, D) = \sum_{|\alpha|\leq 2m} a_{\alpha}(y)D^{\alpha}$ be a strongly elliptic differential operator in Ω.

With suitable boundary conditions, it can be the infinitesimal generator of an analytic semigroup in some function spaces.

The operator

$$A^*(y, D)x = \sum_{|\alpha|\le 2m} (-1)^{|\alpha|} D^{\alpha} (\overline{a_{\alpha}(y)x}) \tag{5.2.1}$$

is called the formal adjoint of A(y, D). From the definition of strong ellipticity it is clear that if A(y, D) is strongly elliptic so is A*(y, D). The coefficients $a_{\alpha}(y)$ of A(y, D) are assumed to be smooth enough, e. g., $a_{\alpha}(y) \in C^{2m}(\overline{\Omega})$ or $C^{\infty}(\overline{\Omega})$.

Case1. $X = L_p(\Omega)$, for $1 < p < \infty$.

Define

$$D(A_p) = W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

and

$$A_p x = A(y, D)x, \text{ for } x \in D(A_p). \tag{5.2.2}$$

The domain $D(A_p)$ of A_p contains $C_0^{\infty}(\Omega)$ and is therefore dense in $L_p(\Omega)$. Then $-A_p$ is the infinitesimal generator of an analytic semigroup T(t), $t \ge 0$ in X (See Pazy, A. (1983)). Therefore the fractional power operator A_p^{α} and the fractional power space X_{α} can be defined.

Case2. $X = L_1(\Omega)$. Define

$$D(A_1) = \{ x \mid x \in W^{2m-1,1}(\Omega) \cap W_0^{m,1}(\Omega),\ A(y, D)x \in L_1(\Omega) \} \tag{5.2.3}$$

where A(y, D)x is in the sense of distributions. For $x \in D(A_1)$, A_1 is defined by

$$A_1 x = A(y, D)x. \tag{5.2.4}$$

$-A_1$ is the infinitesimal generator of an analytic semigroup on $L_1(\Omega)$ (See Pazy, A. (1983)).

Example 2. Let $X = L_2(\Omega)$.

We consider the following controlled problem.

$$\begin{cases} \frac{\partial}{\partial t} x(t,y) + A(y,D)x(t,y) = f_1(t,y,x(t,y),\nabla x(t,y)) + \int_{-r}^{t} h(t-s) f_2(s,y,x(s,y),\nabla x(s,y))ds \\ \qquad\qquad + \int_{\Omega} K(y,\xi)u(\xi,t)d\xi,\ t \in (0,T], \\ x(t,y) = \varphi(t,y),\ y \in \overline{\Omega},\ t \in [-r,0], \\ x(t,y) = 0,\ y \in \partial\Omega,\ t \in [0,T], \end{cases} \tag{5.2.5}$$

where $\Omega \subset R^n$ is a bounded open domain with sufficiently smooth boundary $\partial\Omega$, A(y, D) is the strongly elliptic operator defined as in case 1 of example1, $\varphi \in C^{1,2}([-r,0] \times \overline{\Omega}, R)$, i. e., φ is once continuously differentiable on [-r, 0] and twice continuously differentiable in $\overline{\Omega}$, $u \in L_2(\Omega \times [0,T])$, $h \in L_1([0,T+r],R)$ and $K : \overline{\Omega} \times \overline{\Omega} \to R$ is of Hilbert-Schmidt type, i.e., K is a measurable function such that $\int_{\Omega}\int_{\Omega} |K(y,\xi)|^2 \, dy d\xi < \infty$.

For each $u \in L_2(\Omega\times[0,T])$, let $Bu(t)(y) = \int_\Omega K(y,\xi)u(\xi,t)d\xi$. $B \in L(L_2(I,X))$ is continuous and compact, i. e., B is strongly continuous (See Yosida, K. (1980), pp 277; Renardy, M., Rogers, R. C. (1993), pp. 262-263).

Suppose $f_1 : [0, T]\times \overline{\Omega} \times R\times R^n \to R$ is continuous and there exists constants K_1, $N_1 \geq 0$, a constant $\lambda \geq 1$ such that

$$|f_1(t, y, \xi, \eta)| \leq K_1(1 + |\xi|^\lambda + |\eta|^\lambda),$$

$$|f_1(t, y, \xi_1, \eta_1) - f_1(s, y, , \xi_2, \eta_2)| \leq N_1(|t - s| + |\xi_1 - \xi_2| + |\eta_1 - \eta_2|).$$

We now fix $\frac{3}{4} < \alpha < 1$, $\lambda \in (1, \frac{1}{\alpha})$, we have the imbedding relation $X_\alpha \hookrightarrow C^1(\overline{\Omega})$ (See Amann, H. (1978), pp. 16). Denote the injection by $j_\alpha : X_\alpha \to C^1(\overline{\Omega})$ and define $f : [0, T]\times X_\alpha \to X$ by $f(t, x)(y) = f_1(t, y, j_\alpha(x)(y), \nabla(j_\alpha(x))(y))$. We have

$$\begin{aligned}
\|f(t, x)\|_X &= \|f(t,x)\|_{L_2(\Omega)} \\
&= \left(\int_\Omega |f(t,x)(y)|^2\,dy\right)^{\frac{1}{2}} \\
&= \left(\int_\Omega |f_1(t,y,j_\alpha(x)(y), \nabla j_\alpha(x)(y))|^2\,dy\right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega (K_1(1+|j_\alpha(x)(y)|^\lambda + |\nabla j_\alpha(x)(y)|^\lambda))^2\,dy\right)^{\frac{1}{2}} \\
&= K_1\left(\int_\Omega (1+|j_\alpha(x)(y)|^\lambda + |\nabla j_\alpha(x)(y)|^\lambda)^2\,dy\right)^{\frac{1}{2}} \\
&\leq K_1\left(\int_\Omega (1+\|j_\alpha(x)\|^\lambda_{C^1(\overline{\Omega})})^2\,dy\right)^{\frac{1}{2}} \\
&\leq K_1\left(\int_\Omega (1+c^\lambda\|x\|^\lambda_\alpha)^2\,dy\right)^{\frac{1}{2}} \\
&\leq K_1\left(\int_\Omega dy\right)^{\frac{1}{2}}(1+c^\lambda\|x\|^\lambda_\alpha) \\
&\leq \overline{K}_1(1+\|x\|^\lambda_\alpha),\ \overline{K}_1 = \begin{cases} K_1\left(\int_\Omega dy\right)^{\frac{1}{2}}, \text{if } c^\lambda \leq 1, \\ K_1\left(\int_\Omega dy\right)^{\frac{1}{2}} c^\lambda, \text{if } c^\lambda > 1. \end{cases}
\end{aligned}$$

So we have

$$\|f(t, x)\|_X \leq \overline{K}_1(1+\|x\|^\lambda_\alpha).$$

By a similar argument, we have

$$\begin{aligned}
\|f(t, x_1) - f(t, x_2)\|_X &= \|f(t, x_1) - f(t, x_2)\|_{L_2(\Omega)} \\
&= \Big(\int_\Omega | f(t,x_1)(y) - f(t,x_2)(y)|^2 \, dy\Big)^{\frac{1}{2}} \\
&= \Big(\int_\Omega | f_1(t,y,j_\alpha(x_1)(y), \nabla j_\alpha(x_1)(y)) - f_1(t,y,j_\alpha(x_2)(y), \nabla j_\alpha(x_2)(y))|^2 \, dy\Big)^{\frac{1}{2}} \\
&\le \Big(\int_\Omega \Big[N_1^2 (| j_\alpha(x_1)(y) - j_\alpha(x_2)(y)| + |\nabla j_\alpha(x_1)(y) - \nabla j_\alpha(x_2)(y)|)^2\Big] dy\Big)^{\frac{1}{2}} \\
&= N_1 \Big(\int_\Omega \| j_\alpha(x_1) - j_\alpha(x_2)\|^2_{C_1(\overline{\Omega})} \, dy\Big)^{\frac{1}{2}} \\
&\le N_1 \Big(\int_\Omega (c\| x_1 - x_2 \|_\alpha)^2 dy\Big)^{\frac{1}{2}} \\
&= N_1 c \left(\int_\Omega dy\right)^{\frac{1}{2}} \| x_1 - x_2 \|_\alpha \\
&\le \overline{N}_1 \| x_1 - x_2 \|_\alpha .
\end{aligned}$$

Using a similar procedure to f_1, if $f_2 : [-r, T] \times \overline{\Omega} \times \nabla \times \nabla^n \to \nabla$ is continuous and satisfies:

$$|f_2(t, y, \xi, \eta)| \le K_2(1 + \|\xi\|^\lambda + \|\eta\|^\lambda),$$

$$|f_2(t, y, \xi_1, \eta_1) - f_2(t, y, \xi_2, \eta_2)| \le N_2(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|).$$

Then we can define $g : [-r, T] \times X_\alpha \to X$ by $g(t, x)(y) = f_2(t, y, j_\alpha(x)(y), \nabla j_\alpha(x)(y))$ to have the similar properties:

$$\|g(t, x)\|_X \le \overline{K}_2(1 + \|x\|_\alpha^\lambda),$$

$$\|g(t, x_1) - g(t, x_2)\|_X \le \overline{N}_2 \| x_1 - x_2 \|_\alpha .$$

Now the problem (5.2.5) can be written as

$$\begin{cases} \frac{d}{dt} x(t) + A_p x(t) = f(t, x(t)) + \int_{-r}^{t} h(t-s) g(s, x(s)) ds + Bu(t), t \in (0, T], \\ \qquad x(t) = \varphi(t), t \in [-r, 0]. \end{cases} \tag{5.2.6}$$

Theorem 5.2.1. Suppose the assumptions stated above hold. If there exists a constant $\rho > 0$ such that the a priori estimate $\|x(t,y)\|_{C([0,T]\times\overline{\Omega},R)} \le \rho$ holds, for any possible solution x of the system (5.2.5), then the system (5.2.6) has a unique α - mild solution.

Proof. By using the a priori estimate and applying Theorem 4.1.2, the system (5.2.6) has a unique α -mild solution. □

Remark 5.2.2. If $\lambda = 1$, by using a similar process as in the Global existence theorem (Theorem 3.2.2), Theorem 5.2.1 is still true without assuming the a priori estimate.

We now consider the following cost functional:

$$J(u) = \int_0^T \ell(t, x^u(t), u(t))dt + \psi(x^u(T)),$$

where $\ell : [0, T] \times C^1(\overline{\Omega}) \times L_2(\Omega) \to \nabla \cup \{+\infty\}$, $\psi : L_2(\Omega) \to \nabla$ is defined by

$$\psi(\xi) = \int_\Omega |\xi(y)|^2 \, dy$$

$$\ell(t, x, u) = a(t) \int_\Omega [|x(y)|^2 + |\nabla x(y)|^2]dy + b(t) \int_\Omega |u(y)|^2 \, dy$$

where $a(\cdot), b(\cdot) \in C([0, T]; [0, \infty))$ with min $b(t) = b > 0$.

For each $x \in W^{1,2}(\overline{\Omega})$, $\ell(t, x, u) = a(t)\|x\|_{1,2}^2 + b(t)\|u\|_{L_2(\overline{\Omega})}^2$. By property of the norm and the inequality,

$$\|\alpha x_1 + (1-\alpha)x_2\|_{1,2}^2 - \|\alpha x_1\|_{1,2}^2 - \|(1-\alpha)x_2\|_{1,2}^2 \leq -\alpha(1-\alpha)(\|x_1\|_{1,2} - \|x_2\|_{1,2})^2,$$

$\alpha \in [0, 1]$, it follows that $\ell(t, \bullet, u)$ is convex in $C^1(\overline{\Omega})$ and $\ell(t, x, \bullet)$ is convex in $L_2(\overline{\Omega})$. Since a and b are nonnegative and continuous on [0, T] and the norm is continuous, ℓ is continuous on $[0,T] \times C^1(\overline{\Omega}) \times L_2(\overline{\Omega})$. Since $\ell(t,\bullet,\bullet)$ is continuous and convex on $C^1(\overline{\Omega}) \times L_2(\overline{\Omega})$, then ℓ is weakly sequentially lower semicontinuous on $C^1(\overline{\Omega}) \times L_2(\overline{\Omega})$, (See Zeidler, E. (1990)).Then ℓ is sequentially lower semicontinuous on $C^1(\overline{\Omega}) \times L_2(\overline{\Omega})$.

Similar to the discussion in Theorem 5.2.1 and Remark 5.2.2, applying Theorem 4.2.2 we have the existence of an optimal control as follows.

Theorem 5.2.3. Under the assumptions as in Theorem 5.2.1, there exists a $u^0 \in L_2(\Omega \times [0,T])$ such that $J(u^0) \leq J(u)$, $u \in L_2(\Omega \times [0,T])$.

Chapter VI

Conclusion

We summarize our works into four sections as follows:

6.1 Thesis Summary

In this thesis, we have studied α- mild solutions for a class of semilinear evolution equations whose principal operator is the infinitesimal generator of an analytic semigroup in Banach spaces. We obtained the local existence, global existence, continuous dependence and regularity of mild solutions. A Bolza optimal control problem of a corresponding controlled system can be solved. The application of our abstract results is illustrated by some examples.

We summarize our results:

Part I. Local Existence, Extension and Global Existence

We obtained main theorems as follows:

Theorem 3.1.4. (Local Existence Theorem) Assume that (A), (F1), (G1), and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then there exists a positive number t_1 such that the system (3.1.1) has a unique mild solution on $[-r, t_1]$.

Theorem 3.1.7. (Extension Theorem) Assume (A), (F1), (G1) and (H1) hold.
Let $\varphi \in C([-r,0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Suppose a priori estimate holds for the system (3.1.1), i. e.,

(AP) There exists a constant $\rho > 0$ such that if $x(\bullet)$ is a possible mild solution of the system (3.1.1) on a subset $[-r, T']$ of $[-r, T]$, it follows that $\|x(t)\|_\alpha \le \rho$, for all $t \in [-r, T']$.

Then the system (3.1.1) has a unique global mild solution on [-r, T].

Theorem 3.2.2. (Global Existence Theorem) Assume that (A), (F1), (F5), (G1), (G5) and (H1) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.1.1) has a unique global mild solution on [-r, T].

Theorem 3.2.3. Assume that (A), (F1), (F6), (G1), (G6) and (H1) hold.
Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\lambda\alpha, 1]$. There exists a constant $\rho > 0$ such that if $x(\bullet)$ is a possible mild solution of the system (3.1.1) on a subset $[-r, T']$ of [-r, T], we have

$$\| x(t) \|_E \leq \rho ,$$

for all $t \in [\text{-r}, T']$. Then there exists a constant $\rho^* > 0$ such that

$$\| x(t) \|_\alpha \leq \rho^* ,$$

for all $t \in [\text{-r}, T']$, hence the system (3.1.1) has a unique global mild solution on [-r, T].

Part II. Regularity and Continuous Dependence

Under the same assumptions we can prove that a mild solution is just a classical one. This shows the connection between mild solution and classical solution.

Moreover, we have proved continuous dependence of the system (3.1.1). Some important results of regularity and continuous dependence are as follows:

Theorem 3.3.1. (Regularity) Assume that (A), (F4), (G4), and (H2) hold. Let $\varphi \in C([-r, 0]; X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. If a mild solution x of the system (3.1.9) exists on [-r, T], then $x \in C([-r, T]; X_\alpha) \cap C^1((0, T); X)$, hence it is a classical solution.

Theorem 3.4.1. Assume that the hypotheses of Theorem 3.2.2 are satisfied. For any $\rho > 0$, if x and y are mild solutions of the system (3.1.1) on [-r, T] corresponding to φ_1 and φ_2, respectively, then there exists a constant $K(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \leq K(\rho)\|\varphi_1 - \varphi_2\|_{C([-r, 0]; X_\alpha)},$$

provided $\varphi_1 , \varphi_2 \in C([-r,0]; X_\alpha)$ with $\| \varphi_1 \|_{C([-r,0];X_\alpha)} \leq \rho$ and $\| \varphi_2 \|_{C([-r,0];X_\alpha)} \leq \rho$.

Theorem 3.4.3. Assume that hypotheses of Theorem 3.2.2 are satisfied. For any $\rho > 0$, if x, y are mild solutions of the system (3.1.1) on [-r, T] corresponding to h_1 and h_2, respectively, then there exists a constant $L(\rho) > 0$ such that

$$\|x - y\|_{C([-r,T];X_\alpha)} \leq L(\rho)\|h_1 - h_2\|_{L_1([0,\ T+r];\ L(X))},$$

provided $h_1, h_2 \in L_1([0, T+r]; L(X))$ with $\| h_1 \|_{L_1([0,T+r];L(X))} \leq \rho$ and $\| h_2 \|_{L_1([0,T+r];L(X))} \leq \rho$.

We extended the method of proving global existence to a system with infinite delay and obtained a result as follows:

Theorem 3.5.2. Assume that (A), (F3), (F5), (G5), (G7), (H3) hold. Let $\varphi \in BC((-\infty, 0]; X_\alpha)$ and $,\varphi(0) \in X_\beta$, for some $\beta \in (\alpha, 1]$. Then the system (3.5.1) has a unique mild solution $x \in C((-\infty, T]; X_\alpha)$.

Part III. Existence of Optimal Controls

Existence problem of α - mild solutions of the controlled system (4.1.1) corresponding to the system (3.1.1) can be solved. Existence problem of a more general controlled system is also proved. Existence of an optimal control for a Bolza problem of the system (4.1.1) is presented by using a Balder's result. We obtained main results as follows:

Theorem 4.1.2. Under assumptions (A1), (B), (F2), (F6), (G2), (G6) and (H1).

Let $u \in L_p(I, E)$, $p > \frac{1}{1-\alpha}$, $\varphi \in C([-r,0];X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha,1]$. Then the system (4.1.1) is mildly solvable on $[-r, T]$ with respect to u, and the α–mild solution is unique.

Theorem 4.1.4. Assume that assumptions (A1), (A2), (A3) and (H) hold. Then for each $u \in U_{ad}$ and each $x_0 \in X_\beta$ for some $\beta \in (\alpha,1]$, there exists a constant $t_0 = t_0(u) > 0$ such that the controlled system (4.1.4) is mildly solvable on $[0, t_0]$ with respect to u, and the α–mild solution is unique.

Theorem 4.2.2. Under assumptions (A1), (B), (F2), (G2), (F6), (G6), (H1), (U), (L) and (ψ). Let $\varphi \in C([-r,0];X_\alpha)$ and $\varphi(0) \in X_\beta$, for some $\beta \in (\alpha,1]$. The special Bolza problem (P) has a solution, i. e., there exists an admissible state-control pair $\{u^\circ, x^\circ\}$ such that

$$J(u^\circ) = \int_I \ell(t, x^0(t), u^0(t))dt + \psi(x^0(T)) \leq J(u), \text{ for all } u \in U_{ad}.$$

Part IV Applications

All results in this thesis can be applied to semilinear partial differential equations with delay. Some examples concerning semilinear parabolic differential equations and the corresponding Bolza optimal control problems have been presented.

We also found that

1. Analytic semigroup under fractional power space technique, locally Lipschitz continuity of f and g, and integrability of the function h are important hypotheses for obtaining local existence of mild solutions for the system (3.1.1) and (3.1.9).
2. The a priori estimate is a very important condition that is used to prove the extension of local mild solutions.
3. Gronwall's lemma with singularity and time lag is an important tool for obtaining on a priori estimate and global existence. Moreover, the moment inequality under super linear growth condition gives us a more general theorem of global existence of mild solutions.

6.2 Limitations

1. The infinitesimal generator A we discussed is independent of t.
2. For the optimal control problem, the control part appears linearly in the control system.

3. Necessary conditions for optimality have not been presented.

6.3 Suggestion for Further Work

In the following are important topics that can be studied further.

1. Time optimal control problem and controllability of the systems.
2. Integrodifferential inclusion.
3. Necessary and sufficient condition for optimal controls.
4. System Identification.
5. Stochastic control problems corresponding to our system.
6. Corresponding relaxed controlled system.

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Curriculum Vitae

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